New Families of Integral Sum Graph - Edge Sum Class and Chromatic Integral Sum Number

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Abstract--- Harary [4,5] introduced the concepts of sum and integral sum graphs and a family of integral sum graphs \(G_{n,n}\) over \([-n,n]\) where \([r,s] = \{r,r+1,\ldots,s\}\), \(r \in \mathbb{Z}\). Vilfred [10] generalized \(G_{n,n}\) to integral sum graph \(G_{m,n}\), \(m,n \in \mathbb{N} \cup \{0\}\). An integral sum graph with the label set \(S\) is denoted by \(G^+(S)\). Given an integral sum graph \(G^+(S)\), obtaining an integral sum graph \(G^+(T)\) with \(T \subset S\) or \(T \supset S\) is discussed in this paper. We introduce new families of integral sum graphs, \(G_{X,Y}, H_{m,n}^{XY}, H_{I,J}^{XY}, H_{R,T}^{XY}\). Edges sum class, edge sum color number or chromatic integral sum number \(\chi''(G^+(S))\) of \(G^+(S)\), perfect edge sum color and non-perfect edge sum color graph and obtain a few properties of them. We prove that the set of all non-empty edge sum classes of an integral sum graph partition the edge set of the graph [13].

Keywords--- Integral sum graph, \(G^+(S)\), \(G_n\), \(G_{m,n}\), \(G_{m,n}\) maximal integral sum graph, edge sum class, edge sum color number or chromatic integral sum number \(\chi''(G^+(S))\) of \(G^+(S)\), perfect edge sum color graph and non-perfect edge sum color graph.

I. INTRODUCTION

Harary [4,5] introduced the concepts of sum and integral sum graphs. A graph \(G\) is a sum graph or \(\Sigma\)-graph if the vertices of \(G\) can be labeled with distinct positive integers so that \(e = uv\) is an edge of \(G\) if and only if the sum of the labels on vertices \(u\) and \(v\) is also a label in \(G\). An integral sum graph or \(\Sigma\)-\(Z\)-graph is also defined just as sum graph, the difference being that the labels may be any distinct integers.

Integral sum graph with label set \(S\) is denoted by \(G^+(S)\) and for \(r \in \mathbb{Z}\), \([r,s]\) denotes the set \(\{r,r+1,\ldots,s\}\).

Harary [4,5] introduced a family of sum graphs \(G_n = G^+(\{1,n\})\) and a family of integral sum graphs \(G_{n,n} = G^+(\{-n,n\})\), \(n \in \mathbb{N}\). Vilfred [3] generalized \(G_{n,n}\) to \(G_{m,n}\), \(m,n \in \mathbb{N} \cup \{0\}\). Chen, Harary, Mary Florida, Nicholas, Somasundaram, Suryakala and Vilfred [1,4,17] studied different properties of sum and integral sum graphs. In this paper, we introduce families of integral sum graphs \(G_{X,Y}, H_{I,J}^{m,n}, H_{R,T}^{m,n}\). Define edge sum class, edge sum color number \(\chi''(G^+(S))\) or chromatic integral sum number or chromatic sum number of \(G^+(S)\), classify integral sum graphs into perfect and non-perfect edge sum color graphs and obtain a few properties of them. Open problems on perfect edge sum color graphs and on edge sum color numbers are also given.

For different properties of sum graphs, \(G_{m,n}\), the integral sum graph of order \(n\) with exactly two vertices, each of degree \(n-1, n \geq 4\) and maximal integral sum graphs see [1,3,4,10,16,17]. We consider only simple graphs. \((-G_m) = (-G^+([1,m]))\) denotes integral sum graph \(G^+([-m,-1])\), \(m \in \mathbb{N}\). For all basic ideas in graph theory, we follow [3]. Now, consider some basic definitions and results, which are required in the subsequent sections.

Definition 1.1 [10] Let \(G_{m,n} = G^+([-m,n]), m,n \in \mathbb{N} \cup \{0\}\).

Clearly, \(G_{m,n} = K_1 + (G_m + G_n)\) is an integral sum graph of order \(m+n+1\) and integral sum graphs \(G_{m,n}\) and \(G_{m,n}\) without labels are the same. When we say that \(G_{m,n} \cong G_{m,n}\), then the graphs are considered without the vertex labels. \(G_{1,1} = K_1 + (G_1 + G_1) = K_3\) is the only integral sum graph \(G\) having more than two vertices, each of degree \(\left|V(G)\right| - 1 = 2\). For \(n \geq 2\), in \(G_n\), \(d(u_i) = n-1\) for \(1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\) and \(d(u_i) = n-i\) for \(\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n\) where \(\lfloor x \rfloor\) denotes the floor of \(x\) and \(u_i\) is the vertex with label \(i, 1 \leq i \leq n\). Examples of \(G_{m,n}\) are given in Figures 1 to 4.
sum graph of order \( n \) with the maximum number of edges. Clearly, \( G_n \) is the maximal sum graph of order \( n \), \( n \in \mathbb{N} \).

II. MAIN RESULTS

In this section, we define edge sum class in an integral sum graph, generalize integral sum graph \( G_{m,n} \) to \( G_{X,Y}, H_{i,j}^{XY}, H_{i,j}^{XY}, H_{i,j}^{XY} \) and \( H_{i,j}^{XY} \) and obtained different properties of them.

Without loss of generality, we set \( m \leq n \) for the integral sum graph \( G_{m,n} \), \( m,n \in \mathbb{N} \setminus \{0\} \). Let \( v_i \) denote the vertex whose label is \( i \) in \( G^*(S) \) where \( S = [-m,n] \) and \( m,n \in \mathbb{N} \setminus \{0\} \). In \( G_{m,n} \), let \( [e]_J \) or simply \( [e]_J \) denote the class (set of edges), each with edge sum label \( i \), \( i \in S \). That is \( [e]_J \) in \( G_{m,n} = \{u_p u_q / p+q = i, p,q \in S\} \), \( i \in S \). We call \( [e]_J \) as the edge sum class with edge sum label \( i \), \( i \in S \). Clearly, \( G_{m,n} \setminus \{(u_i) \cup [e_i]\}, G_{m,n} \setminus \{(u_i) \cup [e_i]\} \) and \( G_{m,n} \setminus \{u_i \cup [e_i]\} \) are integral sum graphs, \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

**Definition 2.1** Let \( H_{i,j}^{m,n} = G_{m,n} \setminus \{(u_i) \cup [e_i]\}, H_{i,j}^{m} = G_{m,n} \setminus \{(u_i) \cup [e_i]\} \) and \( H_{i,j}^{m,n} = G_{m,n} \setminus \{u_i \cup [e_i]\} \) and \( H_{i,j}^{m} = G_{m,n} \setminus u_i \) and \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

Clearly, \( H_{i,j}^{m,n} \cong H_{i,j}^{m,n} \cong H_{i,j}^{m,n-1} \cong H_{i,j}^{m,n-1} \cong H_{i,j}^{m,n-1} \cong H_{i,j}^{m,n} \cong H_{i,j}^{m,n} \cong H_{i,j}^{m,n} \cong H_{i,j}^{m,n} \) and \( H_{i,j}^{m,n} \) and \( H_{i,j}^{m,n} \) are integral sum graphs with vertex sets \( S = [-m,n] \) \{0\}, \( S = [-m,n] \) \{1\}, \( S = [-m,n] \) \{0,1\}, \( S = [-m,n] \) \{i\}, \( S = [-m,n] \) \{i,j\} and \( S = [-m,n] \) \{i,j,k\}, respectively. \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

In general, \( H_{X,Y}^{m,n} = G_{m,n} \setminus \{(X \cup Y \cup \{X\}(X \cup Y) \} \) where \( X \subseteq \{1,m\}, y \subseteq \{1,n\} \) and \( X = \cup_{i \in S} \{e_i\} \) and \( Y = \cup_{j \in Y} \{e_j\} \). Clearly, \( H_{X,Y}^{m,n} \) and \( H_{Y,X}^{m,n} \) are integral sum graphs and \( H_{X,Y}^{m,n} \cong H_{Y,X}^{m,n} \) without the vertex labels, \( m,n \in \mathbb{N} \).

For example, in \( G_{2,4}, [e_0] = \{(1,1),(-2,2)\}, [e_1] = \{(-1,0),(-2,1)\}, [e_2] = \{(0,1),(-1,2),(-2,3)\}, [e_3] = \{(2,0),(-1,3),(-2,4)\}, [e_4] = \{(0,3),(-1,4)\} \) and \( [e_i] = \{(0,4),(1,3)\} \). See Figure 3.

**Theorem 2.2** For \( m,n \in \mathbb{N} \),

(i) \( |E(G_{m,n}^+)\| = |E(G_{m,n})|-(m+n)-\min\{m,n\} \);

(ii) \( |E(H_{i,j}^{m,n})^+\| = |E(G_{m,n})|-(m+n+i)-\min\{m,n\} \) if \( 1 \leq i \leq \min\{m,n\} \) and \( \min\{m+i,n+1\} \) if \( \min\{m,n\} < i \leq m \);

(iii) \( |E(H_{i,j}^{m,n})^+\| = |E(G_{m,n})|-(m+n+i)-\min\{m,n\} \) if \( 1 \leq i \leq \min\{m,n\} \) and \( \min\{m+i,n+1\} \) if \( \min\{m,n\} < i \leq m \).

**Proof** (i) The result follows from \( H_{i,j}^{m,n} = G_{m,n} \setminus \{(u_i) \cup [e_i]\} \) \( = G_{m,n} \setminus \{u_i (0,0),(1,0),(0,1),(1,0)\} \) \{k,k\} \( i = 1,2,....m; j = 1,2,....n \) and \( k = 1,2,....\\min\{m,n\} \).

(ii) We have \( H_{i,j}^{m,n} = G_{m,n} \setminus \{(u_i) \cup [e_i]\}, 1 \leq i \leq m \).

Thus, \( H_{i,j}^{m,n} \) is obtained from \( G_{m,n} \) by removing vertex \( u_i \) and edge sum class \( [e_i] \). Let \( k = \min\{m,n\}, 1 \leq i \leq m \).

Given an integral sum graph \( G_{m,n} \) of order \( n \), how to find out the maximal integral sum subgraph(s) of order \( n \)? \( H_{0}^{m,n} \), \( H_{1,i}^{m,n} \) and \( H_{i,j}^{m,n} \) are integral sum graphs, each of order \( n \) and we obtain them by simply removing one vertex with label 0, -1 or i and corresponding edges, each with edge sum label 0, -1 or i from the respective graph. By considering \( |E(H_{0}^{m,n})| \), \( |E(H_{1,i}^{m,n})| \) and \( |E(H_{i,j}^{m,n})| \) for all possible values of \( i \), it is possible to find out the maximal integral sum subgraph(s) of order \( n \), if exists.
Theorem 2.4 For a given natural number n and for different possible values of i and r, integral sum graph $H_{i,r}^{n,m}$ has maximum number of edges when i = r, 1 ≤ i ≤ r.

Proof For the sake of simplicity, let r ≤ n, r,n-r∈N. Then, for 1 ≤ i ≤ r, using Theorem 2.2,

$$|E(H_{i,r}^{n,m})| = |E(G_{n-r})|-(r+n-2i)-\left\lfloor \frac{r}{2} \right\rfloor$$

if 1 ≤ i ≤ \left\lfloor \frac{r}{2} \right\rfloor and

$$= |E(G_{n-r})|-(r+n-2i)-\left\lceil \frac{r}{2} \right\rceil$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r.

Let x = \left\lfloor \frac{r}{2} \right\rfloor, r,n-r∈N. Using Theorem 1.2, we get,

$$|E(G_{r,n-r})| = \frac{1}{4}(r^2+(n-r)^2+3n+4r(n-r)-x)(x-2), r,n-r∈N∪\{0\}.$$

Thus, we get,

$$|E(H_{r,n}^{n,m})| = \frac{1}{4}(r^2+(n-r)^2+3n+4r(n-r)-x)(x-2)$$

if 1 ≤ i ≤ \left\lfloor \frac{r}{2} \right\rfloor and

$$= \frac{1}{4}(r^2+(n-r)^2+3n+4r(n-r)-x)(x-2)$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r.

That is $|E(H_{i,r}^{n,m})| = \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lfloor \frac{r}{2} \right\rfloor$ if 1 ≤ i ≤ \left\lceil \frac{r}{2} \right\rceil and

$$= \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lceil \frac{r}{2} \right\rceil$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r.

For further simplification of $|E(H_{i,r}^{n,m})|$, we consider the above 4 cases.

Case 1 r and n are even.

Let r = 2k and n = 2m, k,m∈N. And in this case, $\left\lfloor \frac{r}{2} \right\rfloor = k+(m-k) = m/2, m∈N$. This implies, $|E(H_{i,r}^{n,m})| = \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lfloor \frac{r}{2} \right\rfloor$ if 1 ≤ i ≤ \left\lfloor \frac{r}{2} \right\rfloor and

$$= \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lceil \frac{r}{2} \right\rceil$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r, r,n-r∈N. Hence the result.

That is $|E(H_{n}^{n,m})| = \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)$ if 1 ≤ i ≤ \left\lceil \frac{r}{2} \right\rceil and

$$= \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r, r,n-r∈N.

Case 2 r is even and n is odd.

Let r = 2k and n = 2m+1, k,m∈N. In this case, $\left\lfloor \frac{r}{2} \right\rfloor = k+(m-k) = m/2, m∈N$. This implies, $|E(H_{i,r}^{n,m})| = \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lfloor \frac{r}{2} \right\rfloor$ if 1 ≤ i ≤ \left\lfloor \frac{r}{2} \right\rfloor and

$$= \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lceil \frac{r}{2} \right\rceil$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r, r,n-r∈N. Hence the result. From the above discussion, we obtain for 1 ≤ i ≤ r and for all possible values of i, the maximum value of $|E(H_{i,r}^{n,m})|$ is $|E(H_{r,n}^{r,n})| = |E(G_{r-1,n-r})|$, r ≥ n, r,n-r∈N. Hence the result. For 1 ≤ i ≤ r and r,n-r∈N, using Theorems 1.2 and 2.2, $|E(H_{i,r}^{n,m})| = |E(G_{r-1,n-r})|$, r,n-r∈N.

Case 3 r is odd and n is even.

Let r = 2k+1 and n = 2m, k,m∈N. In this case, $\left\lfloor \frac{r}{2} \right\rfloor = k+(m-k) = m/2, m∈N$. This implies, $|E(H_{i,r}^{n,m})| = \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lfloor \frac{r}{2} \right\rfloor$ if 1 ≤ i ≤ \left\lfloor \frac{r}{2} \right\rfloor and

$$= \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lceil \frac{r}{2} \right\rceil$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r, r,n-r∈N. Hence the result. From the above discussion, we obtain for 1 ≤ i ≤ r and for all possible values of i, the maximum value of $|E(H_{i,r}^{n,m})|$ is $|E(H_{r,n}^{r,n})| = |E(G_{r-1,n-r})|$, r ≥ n, r,n-r∈N. Hence the result. For 1 ≤ i ≤ r and r,n-r∈N, using Theorems 1.2 and 2.2, $|E(H_{i,r}^{n,m})| = |E(G_{r-1,n-r})|$, r,n-r∈N.

Case 4 r and n are odd.

Let r = 2k+1 and n = 2m+1, k,m∈N. In this case, $\left\lfloor \frac{r}{2} \right\rfloor = k+(m-k) = m/2, m∈N$. This implies, $|E(H_{i,r}^{n,m})| = \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lfloor \frac{r}{2} \right\rfloor$ if 1 ≤ i ≤ \left\lfloor \frac{r}{2} \right\rfloor and

$$= \frac{1}{4}(r^2+(n-r)^2-2n-4r(n-r)+x)(x-2)\left\lceil \frac{r}{2} \right\rceil$$

if \left\lceil \frac{r}{2} \right\rceil < i ≤ r, r,n-r∈N. Hence the result. From the above discussion, we obtain for 1 ≤ i ≤ r and for all possible values of i, the maximum value of $|E(H_{i,r}^{n,m})|$ is $|E(H_{r,n}^{r,n})| = |E(G_{r-1,n-r})|$, r ≥ n, r,n-r∈N. Hence the result. For 1 ≤ i ≤ r and r,n-r∈N, using Theorems 1.2 and 2.2, $|E(H_{i,r}^{n,m})| = |E(G_{r-1,n-r})|$, r,n-r∈N.

Proceed Consider, integral sum graph $G_{m,n}$ of order m+n+1, m,n∈N. Using definition 2.1 of $H_{i,r}^{m,n}$, we get,

$$|E(H_{m,n}^{m,n})| = |E(H_{i,r}^{m,n})| \cdot |E(G_{m,n})| + \text{ the number of edges that are removed twice, once to get } H_{i,r}^{m,n} \text{ from } G_{m,n} \text{ and the other time to get } H_{i,r}^{m,n} \text{ from } G_{m,n}.$$

For 1 ≤ i ≤ m and 1 ≤ j ≤ n, (-i,j) is an edge in $G_{m,n}$. Similarly, (-i,j)∈E(G_{m,n}) if i+j ≤ m and (-i,j)∈E(G_{m,n}) if i+j ≤ n. These are the possible edges (if they occur) which are removed twice, once to get $H_{i,r}^{m,n}$ from $G_{m,n}$ and the other time to get $H_{i,r}^{m,n}$ from $G_{m,n}$.

Also, using Theorems 1.2 and 2.2, we get, $|E(G_{m,n})| = \frac{1}{4}(m^2+n^2+3(m+n)+4mn)-\frac{1}{2}(m^2+n^2)/2$, m,n∈N∪\{0\};

$$|E(H_{i,r}^{m,n})| = |E(G_{m,n})| \cdot |E(G_{m,n})| \cdot \left\lfloor \frac{r}{2} \right\rceil$$

if 1 ≤ i ≤ \left\lfloor \frac{r}{2} \right\rceil and
\[ |E(G_{m,n})| \leq (2m+n+1-2i)-\lfloor \frac{(i-1)}{2} \rfloor \text{ if } \lfloor \frac{(i-1)}{2} \rfloor + 1 \leq i \leq m. \]

And correspondingly, we consider the following four cases. In each case, one figure is given corresponding to a subgraph of \( G_{m,n} \) that contains vertices with label 0, -i, j, -i-j, i+j, -m, n. In each figure, we consider vertices with label 0, -i, j, -i-j, i+j, -m, n and their corresponding edges and not the whole subgraph. Dotted line, in these figures, denotes edge which occurs in both the graphs \( H_{i,j} \) and \( H_{j,i} \). Isolated vertex or vertices in these figures indicate that the corresponding subgraph of \( G_{m,n} \) does not contain these isolated vertex (or vertices) using the definition of integral sum graph. For the sake of simplicity, let \( k = \lfloor \frac{(i-1)}{2} \rfloor + \lfloor \frac{(j-1)}{2} \rfloor, \) \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) and we set \( m \leq n \) for the integral sum graph \( G_{m,n} \).

**Case 1**

\[ 1 \leq i \leq \lfloor \frac{m}{2} \rfloor \text{ and } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor. \]

In this, \( 1 \leq i+j \leq n \) and the following two subcases arise.

**Case 1.1**

\[ 1 \leq i+j \leq m. \]

In this case, \( i+j = m \). See Figure 8. This implies, \[ |E(H_{i,j}^{m,n})| = |E(H_{j,i}^{m,n})| + |E(G_{m,n})| + 3 = |E(G_{m,n})|(2m+n-2i)-(m+2n-2j)+3-k = |E(G_{m,n})|(3m+3n+2-2i-2j)+3-k \]

This implies, \[ |E(H_{j,i}^{m,n})| = |E(H_{j,i}^{m,n})| + |E(H_{i,j}^{m,n})| + |E(G_{m,n})| + 3 = |E(G_{m,n})|(2m+n-2i)-(m+2n-2j)+3-k = |E(G_{m,n})|(3m+3n+2-2i-2j)+3-k \]

\[ = \frac{1}{4}(m^2+n^2-9(m+n)+4mn)-\frac{1}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 2(i+j) + k. \]

**Case 2**

\[ m+1 \leq i+j \leq n. \]

In this case, \( i+j = n \). See Figure 9. This implies, \[ |E(H_{i,j}^{m,n})| = |E(H_{j,i}^{m,n})| + |E(G_{m,n})| + 3 = |E(G_{m,n})|(2m+n-2i)-(m+2n+1-2j)+3-k = |E(G_{m,n})|(3m+3n+2-2i-2j)+3-k \]

\[ = \frac{1}{4}(m^2+n^2-9(m+n)+4mn)-\frac{1}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 2(i+j) + 1-k. \]

**Case 3**

\[ m+1 \leq i \leq m \text{ and } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor. \]

Under this, the following three subcases arise.

**Case 3.1**

\[ m+2 \leq i+j \leq m. \]

Here, \[ |E(H_{i,j}^{m,n})| = |E(H_{j,i}^{m,n})| + |E(G_{m,n})| + 3 = |E(G_{m,n})|(2m+n+1-2i)-(m+2n-2j)+3-k = |E(G_{m,n})|(3m+3n+2-2i-2j)+2-k \]

\[ = \frac{1}{4}(m^2+n^2-9(m+n)+4mn)-\frac{1}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 2(i+j) + 2-k. \]

**Case 3.2**

\[ m+1 \leq i+j \leq n. \]

Under this, \[ |E(H_{i,j}^{m,n})| = \frac{1}{4}(m^2+n^2-9(m+n)+4mn)-\frac{1}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 2(i+j) + k. \]

**Case 3.3**

\[ n+1 \leq i+j \leq n. \]

Here, \[ |E(H_{i,j}^{m,n})| = |E(H_{j,i}^{m,n})| + |E(G_{m,n})| + 3 = |E(G_{m,n})|(2m+n+1-2i)-(m+2n+1-2j)+2-k = |E(G_{m,n})|(3m+3n+2-2i-2j)+1-k \]

\[ = \frac{1}{4}(m^2+n^2-9(m+n)+4mn)-\frac{1}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 2(i+j) + 1-k. \]

**Case 4**

\[ 1 \leq i+j \leq n. \]

Here, \( i+j \geq m+2 \) and the following two subcases arise.

**Case 4.1**

\[ m+1 \leq i+j \leq n. \]

Here, \( i+j = m+1 \). See Figure 15. This implies, \[ |E(H_{i,j}^{m,n})| = |E(H_{j,i}^{m,n})| + |E(G_{m,n})| + 3 = |E(G_{m,n})|(2m+n+1-2i)-(m+2n+1-2j)+2-k = \frac{1}{4}(m^2+n^2-9(m+n)+4mn)-\frac{1}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 2(i+j) - k. \]

**Case 4.2**

\[ i+j = m+1. \]

In this case, \( i+j = m+1 \). See Figure 17. This implies, \[ |E(H_{i,j}^{m,n})| = |E(G_{m,n})|(3m+3n+2-2i-2j)+1-k = \frac{1}{4}(m^2+n^2-9(m+n)+4mn)-\frac{1}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 2(i+j) + 1-k. \]

From all the above possible cases, it is clear that for given natural numbers \( m \) and \( n \), \[ |E(H_{i,j}^{m,n})| \text{ is maximum when } 2(i+j)-\lfloor \frac{(i-1)}{2} \rfloor-\lfloor \frac{(j-1)}{2} \rfloor \text{ is maximum}. \]

For \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), \( 2i+2j-\lfloor \frac{(i-1)}{2} \rfloor-\lfloor \frac{(j-1)}{2} \rfloor \) is maximum when i and j have maximum values and the possible maximum values of i and j are m and n, respectively. Thus, for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), the maximum value of \[ |E(H_{i,j}^{m,n})| \text{ is } |E(H_{1,1}^{m,n})| = |E(G_{m-1,n-1})|. \] Hence the result.
So far, we obtained integral sum graphs whose order is less than the order of a given integral sum graph by removal of one or two vertex labels and the corresponding edge sum classes. To obtain integral sum graph of any smaller size from $G_{m,n}$, we consider graph $H_{R,T}^{X,Y}$ by removing a set of vertex labels and the corresponding edge sum classes where $R \subseteq X \subseteq N$ and $T \subseteq Y \subseteq N$. Here, $H_{R,T}^{X,Y} = G_{X,Y} = G_{S}$ where $S = \{0\} \cup (-X) \cup Y$ and $X,Y \subseteq N$. 

III. PROPERTIES OF EDGE SUM CLASSES

A few properties of the set of all edge sum class of any integral sum graph $G^{*}(S)$ is obtained in this section.

Theorem 3.1 Let $G^{*}(S)$ be an integral sum graph of order $n$, $n \in N$. Then, (i) $E(G^{*}(S)) = U_{i \in S} [E_{i}]$. (ii) Two edge sum classes are either equal or disjoint. That is for every $i,j \in S$, either $[E_{i}] = [E_{j}]$ or $[E_{i}] \cap [E_{j}] = \phi$. (iii) For $i,j \in S$ if $[E_{i}] \neq \phi$, $[E_{j}] \neq \phi$ and $i \neq j$, then $[E_{i}] \neq [E_{j}]$. (iv) A non-empty edge sum class is an independent set of edges of the integral sum graph. That is no two elements (edges) of an edge sum class have common vertex. (v) The number of distinct non-empty edge sum classes of $G^{*}(S)$ is less than or equal to $n$, the order of the graph. (vi) The number of distinct non-empty edge sum classes of $G^{*}(S)$ is equal to $n$ if and only if every vertex label occurs as the induced sum of at least one edge in $G^{*}(S)$.

Proof Properties (i), (ii) and (iii) follow from the definition of edge sum class. Property (iv) follows from the definition of integer sum labeling. Properties (v) and (vi) follow from the definition of integer sum labeling and from the fact that each edge belongs to an edge sum class, say, $[E_{i}]$, $i \in S$.

Theorem 3.2 The set of all non-empty edge sum classes of an integral sum graph $G^{*}(S)$ partition the set of all edges of the graph.

Proof The result follows from properties (i), (ii) and (iii) of Theorem 3.1.

We have seen that the set of all non-empty edge sum classes of an integral sum graph partition the edge set of the graph. For a given integral sum graph $G^{*}(S)$, the set of all edge sum classes is unique. Property (iv) helps us to consider an integral sum graph as an edge sum color graph by applying same color to all edges in an edge sum class and different colors to different edge sum classes.

For a given integral sum graph or edge sum color graph $G^{*}(S)$, the edge coloring of $G^{*}(S)$ obtained by considering each edge sum class as an edge color class need not be a minimal edge coloring of $G$. For example, consider the integral sum graph $G_{1,5}$. See Figure 2. The edge sum classes of $G_{1,5}$ are $[-1] = \{(0,-1)\}$, $[0] = \{(-1,1)\}$, $[1] = \{(0,1), (-1,2)\}$, $[2] = \{(0,2), (-1,3)\}$, $[3] = \{(-1,4), (0,3), (1,2)\}$, $[4] = \{(-1,5), (0,4), (1,3)\}$, $[5] = \{(0,5), (1,4), (2,3)\}$. On the other hand, one set of edge color classes of $G_{1,5}$ are $\{((-1,2), (0,1)), (-1,3), (0,2)\}$, $\{(-1,4), (0,3), (1,2)\}$, $\{(-1,5), (0,4), (1,3)\}$, $\{(-1,1), (0,5)\}$. Thus, there are seven edge sum classes in $G_{1,5}$ whereas its edge color number is six.

Definition 3.3 Integral sum graph $G^{*}(S)$ is said to be a perfect edge sum color graph if the edge sum classes are also minimal edge color classes of $G^{*}(S)$. Otherwise, $G^{*}(S)$ is called a non-perfect edge sum color graph.

We have seen that $G_{1,5}$ is a non-perfect edge sum color graph. Star graphs are perfect edge sum color graphs. In Problem 3.5, we prove that for $n \in N$, $G_{1,n}$ and $G_{0,n}$ are perfect edge sum color graphs and for $2 \leq n$, $G_{1,n}$ is a non-perfect edge sum color graph.

Definition 3.4 For non-empty distinct edge sum classes of an integral sum graph $G^{*}(S)$ is called the edge sum color number or chromatic edge sum number or integral sum number or chromatic sum number of $G^{*}(S)$ and is denoted by $\chi'(G^{*}(S))$.

It is clear that for a given integral sum graph $G^{*}(S)$, $\chi'(G^{*}(S)) = \text{order of the graph } G^{*}(S)$ if and only if $[E_{i}] \neq \phi$ for every $i \in S$.

Problem 3.5 Prove that (i) for $m,n \in N$, $\chi'(G_{m,n}) = m+n+1$, (ii) for $n \in N$, $G_{1,n}$ and $G_{0,n}$ are perfect edge sum color graphs, (iii) for $2 \leq n$, $G_{1,n}$ is a perfect edge sum color graph and (iv) for $2 \leq n \leq 6$, $G_{0,n}$ is a non-perfect edge sum color graph.

Solution (i) The edge sum classes of the integral sum graph $G_{m,n}$ are $[-m], [-m+1], \ldots, [0], [1], \ldots, [n]$ and each one is non-empty. Hence, $\chi'(G_{m,n}) = m+n+1$ for $m,n \in N$.

(ii) $\chi'(G_{1,1}) = 3 = \chi'(G_{1,1})$. Now, consider $G_{0,n}$. $\chi'(G_{0,n})$ is a perfect edge sum color graph. The edge sum classes of integral sum graph $G_{0,n}$ are $[0], [1], \ldots, [n]$ and each one, except $[0]$, is non-empty and hence $\chi'(G_{0,n}) = n$. Also, for any graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, Vizing’s Theorem [4]. Now, $\{ (0,1) \}, \{ (0,2) \}, \{ (0,3), (1,2) \}, \{ (0,4), (1,3) \}, \{ (0,5), (1,4), (2,3) \}, \{ (0,6), (1,5), (2,4) \}, \ldots \ldots \{ (0,n-1), (1,n-1), (2,2n-2), \ldots \ldots \{ (n-1,2), (n-2,2(n-2)), \ldots \ldots (n,2) \})$ is a set of edge color classes of $G_{0,n}$. This implies, $\chi'(G_{0,n}) = n = \Delta(G_{0,n}) = \chi'(G_{0,n})$ and hence $G_{0,n}$ is a perfect edge sum color graph for $n \in N$.

(iii) For $2 \leq n$, the edge sum classes of integral sum graph $G_{1,n}$ are $[0], [1], \ldots, [n]$, each one is non-empty and hence $\chi'(G_{1,n}) = n+2$. $\Delta(G_{1,n}) = n+1$ degree of the vertex whose label is 0 in $G_{1,n}$ and $\{ (0,1), (-1,2) \}, \{ (0,2), (-1,3) \}, \{ (0,3), (-1,4), (1,2) \}, \{ (0,4), (-1,5), (1,3) \}, \{ (0,5), (-1,6), (1,4), (2,3) \}, \{ (0,6), (-1,7), (1,5), (2,4) \}, \{ (0,7), (-1,8), (1,6), (2,5), (3,4) \}, \{ (0,8), (-1,9), (1,7), (2,6), (3,5) \}, \{ (0,9), (-1,10) \}, \{ (1,10), (2,2n-3), \ldots \ldots \{ (n-1,2), (n-2,2(n-2)), \ldots \ldots (n,2) \})$ is a set of edge color classes of $G_{1,n}$ which is of order $n+1$. This implies, edge color number of $G_{1,n}$ is $n+1$, using Vizing’s Theorem [4]. This implies, for $2 \leq n$, $\chi'(G_{1,n}) = 1+n \neq \chi'(G_{1,n}) = n+2$ and thereby, $G_{1,n}$ is a non-perfect edge sum color graph.

(iv) At first, a set of edge color classes of $G_{n,m}$ each of order $2n$ for $n = 2,3,4,5,6$ is obtained as follows:

(a) A set of edge color classes of order 4 of $G_{2,2}$ is $\{ (0,2), (-1,1) \}, \{ (0,1), (-2,2) \}, \{ (0,-1), (-2,1) \}, \{ (0,2), (-1,2) \}$.

(b) A set of edge color classes of order 6 of $G_{3,3}$ is $\{ (0,3), (1,2) \}, \{ (0,2), (-3,1) \}, \{ (0,1), (-2,3), (-1,2) \}, \{ (0,1), (-3,2), (-2,1) \}, \{ (0,2), (-1,3) \}, \{ (0,3), (-2,1) \}$.
(c) A set of edge color classes of order 8 of \( G_{4,4} \) is \{\{(0,4), (-3,4), (2,2), (-1,1)\}, \{(0,3), (-4,4), (1,2)\}, \{(0,2),(-4,1), (-3,3), (2,4), (-1,2)\}, \{(0,1),(-3,3), (2,4), (-1,2)\}, \{(0,1),(-3,3), (2,4), (-1,2)\}\}

(d) A set of edge color classes of order 10 of \( G_{5,5} \) is \{\{(0,5), (-5,1), (-4,1), (-3,2), (2,3)\}, \{(0,4), (-5,3), (-4,1), (-3,2), (2,3)\}, \{(0,3), (-5,1), (-4,2), (-3,2), (2,3)\}, \{(0,2),(-5,1), (-4,2), (-3,2), (2,3)\}, \{(0,1),(-5,1), (-4,2), (-3,2), (2,3)\}\}

(e) A set of edge color classes of order 12 of \( G_{6,6} \) is \{\{(0,6), (-6,1), (-4,2), (-2,6), (-1,5), (0,3), (-6,6), (-4,5), (-2,1), (2,3)\}, \{(0,5), (-6,1), (-4,2), (-2,6), (-1,5), (0,3), (-6,6), (-4,5), (-2,1)\}, \{(0,4), (-6,1), (-4,2), (-2,6), (-1,5), (0,3), (-6,6), (-4,5), (-2,1)\}, \{(0,3), (-6,1), (-4,2), (-2,6), (-1,5), (0,3), (-6,6), (-4,5), (-2,1)\}\}

Using (i), \( \chi'(G_{m,n}) = mn+1 \) for \( m,n \in \mathbb{N} \) and thereby \( \chi'(G_{m,n}) = 2n+1 \) for \( n \in \mathbb{N} \). For \( 2 \leq n \leq 2 \), using Vizing’s Theorem \([4]\), edge color number of \( G_{m,n} \) is 2. Thus, for \( 2 \leq n \leq 2 \), \( \chi'(G_{m,n}) = 2n \neq \chi'(G_{n,n}) = 2n+1 \) and thereby, \( G_{m,n} \) is a non-perfect edge sum color graph. Hence the result.

Conjecture 3.6 For \( 3 \leq m+n \) and \( m,n \in \mathbb{N} \), \( \chi'(G_{m,n}) = mn \) and \( G_{m,n} \) is a non-perfect edge sum color graph. In particular, \( \chi'(G_{m,n}) = 2n \) and \( G_{m,n} \) is a non-perfect edge sum color graph for \( 2 \leq n \leq n \).

Open problems 3.7 Characterize integral sum graphs \( G'(S) \) such that (i) \( \chi'(G'(S)) = \) order of the graph \( G'(S) \); (ii) \( G'(S) \) is a a perfect edge sum color graph; (ii) \( G'(S) \) is a non-perfect edge sum color graph; (iii) \( \chi'(G'(S)) = \chi'(G'(S)) \); (iv) \( \chi'(G'(S)) < \chi'(G'(S)) \) and (v) \( \chi'(G'(S)) > \chi'(G'(S)) \).

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