Self Vertex Switching of Connected Bicyclic Graphs

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Abstract—A vertex \( v \) \( V(G) \) is said to be a self vertex switching of \( G \) if \( G \) is isomorphic to \( G' \), where \( G' \) is the graph obtained from \( G \) by deleting all edges of \( G \) incident to \( v \) and adding all edges incident to \( v \) which are not in \( G \). In [5], trees and forests are characterized, each with a self vertex switching. In [6], connected unicyclic graphs, each with a self vertex switching are characterized. In this paper, we characterize connected bicyclic graphs, each with a self vertex switching.

Keywords---Switching, Self Vertex Switching, Bicyclic, \( SS(G) \), \( ss\_s(G) \).

I. INTRODUCTION

For a finite undirected simple graph \( G(V,E) \) with \( |V(G)| = p \) and a set \( \sigma \subseteq V \), the switching of \( G \) by \( \sigma \) is defined as the graph \( G'\sigma(V,E\sigma) \), which is obtained from \( G \) by removing all edges between \( \sigma \) and its complement \( V - \sigma \) and adding as edges all non-edges between \( \sigma \) and \( V-\sigma \). Switching has been defined by Seidel [2] and is also referred to as Seidel switching. When \( \sigma = \{v\} \subseteq V \), we call the corresponding switching \( G^{(v)} \) as vertex switching and denoted it as \( G' \) [1]. A subset \( \sigma \) of \( V(G) \) to be a self switching of \( G \) if \( G \cong G' \). The set of all self switching of \( G \) with cardinality \( k \) is denoted by \( SS_k(G) \) and its cardinality by \( ss\_s(G) \). If \( k = 1 \), then we call the corresponding self switching as self vertex switching [1, 3].

A branch at \( v \) in \( G \) is a maximal connected subgraph \( B \) of \( G \) such that the intersection of \( B \) with the vertex \( v \) is \( v \) and \( B - v \) is connected [3]. A walk in a graph is a finite non-null sequence whose terms are alternatively vertices and edges. A path is a walk in which all the vertices are distinct. A path with \( n \) vertices is denoted by \( P_n \). Two vertices \( u \) and \( v \) are said to be interchange similar if there is an automorphism \( f \) of \( G \) such that \( f(u) = v \) and \( f(v) = u \).

In [4], a characterization of interchange similar vertices to be self vertex switchings was given. In [5], trees and forests are characterized, each with a self vertex switching. In [6], connected unicyclic graphs, each with a self vertex switching is characterized. In this paper we characterize the connected bicyclic graphs, each with a self vertex switching. We consider simple graphs only. Now consider the following results, which are required in the subsequent sections.

Theorem 1.1.[1] If \( v \) is a self vertex switching of a graph \( G \) of order \( p \), then \( d_G(v) = (p-1)/2 \).

Theorem 1.2.[3] Let \( v \) be any vertex of a connected graph \( G \) such that \( G' \) is connected. Then \( B \) is a branch at \( v \) in \( G \) if and only if \( B' \) is a branch at \( v \) in \( G' \).

Theorem 1.3.[5] Let \( v \in V(G) \) and \( |V(G)| \geq 2 \). Then \( G' \) is connected if and only if \( d_G(v) = 0 \) or \( d_G(v) < |V(B)|-1 \) for every branch \( B \) at \( v \) in \( G \).

Theorem 1.4.[5] Let \( v \) be a vertex of a non-trivial connected graph \( G \). Then \( G' \) is a tree if and only if \( G-v \) is acyclic and \( d_G(v) = |V(B)|-2 \) for every branch \( B \) at \( v \) in \( G \).

Theorem 1.5.[5] \( D \) is a component of \( G \) not containing \( v \) if and only if \( D+v \) is a branch at \( v \) in \( G' \).

Theorem 1.6.[5] Let \( v \) be a self vertex switching of a connected graph \( G \) and let \( B \) be a branch at \( v \) in \( G \). Then \( |V(B)| \geq 3 \).

Theorem 1.7.[6] Let \( v \in V(G) \) be a non-cut vertex of \( G \) of order \( p \geq 3 \). Then \( G' \) is connected and unicyclic if and only if either of the following holds:

a. \( G = K_2U(p-2)K_1 \) and \( v \) is one of the \( K_1 \)'s.

b. \( G \) is connected, \( G-v \) is acyclic and \( d_G(v) = |V(G)|-3 \).

c. \( G \) is connected, \( G-v \) is unicyclic and \( d_G(v) = |V(G)|-2 \).

d. \( G = DU(p-|V(D)|)K_1 \), \( G-v \) is unicyclic and \( d_G(v) = |V(D)|-2 \).

e. \( G = DU(p-|V(D)|)K_1 \), \( G-v \) is acyclic and \( d_G(v) = |V(D)|-3 \).

f. \( G = DUK_2U(p-2)K_1 \), \( G-v \) is acyclic and \( d_G(v) = |V(D)|-2 \) where \( D \neq K_1 \), \( K_2 \) is a component of \( G \) containing \( v \).

Theorem 1.8.[6] Let \( v \) be a cut vertex of a graph \( G \) of order \( p \geq 3 \). Then \( G' \) is connected and unicyclic if and only if either of the following holds:

a. \( G \) is connected, \( G-v \) is acyclic, \( d_G(v) \in \{ |V(B)|-2, |V(B)|-3 \} \) for any branch \( B \) at \( v \) in \( G \) and \( d_G(v) = |V(B)|-3 \) only for one \( B \).

b. \( G \) is connected, \( G-v \) is unicyclic, \( d_G(v) = |V(B)|-2 \), for any branch \( B \) at \( v \) in \( G \) and \( B-v \) is unicyclic for one \( B \).
c. $G = DU_KU(p-2\lfloor V(D)\rfloor)K_1$, $G-v$ is acyclic and $d_0(v) = |V(B)|-2$ for any branch $B$ at $v$ in $D$. \\
d. $G = DU[p-\lfloor V(D)\rfloor]K_1$, $G-v$ is acyclic and $d_0(v) \in \{V(B)|-2, \lfloor V(B)|-3\}$ for any branch $B$ at $v$ in $D$ and $d_0(v) = |V(B)|-3$ only for one $B$. \\
e. $G = DU[p-\lfloor V(D)\rfloor]K_1$, $G-v$ is unicyclic, $d_0(v) = |V(B)|-2$ for any branch $B$ at $v$ in $D$ and $B-v$ is unicyclic for one $B$, where $D \neq K_1$, $K_2$ is a component of $G$ containing $v$.

II. CHARACTERIZATION OF $G_V$ TO BE CONNECTED AND BICYCLIC

Let $v$ be a vertex of a graph $G$. Let $G'$ be the switching of $G$ by $v$. In [5], we gave a condition on vertex $v$ of $G$ such that $G'$ is connected. In [6], we characterize vertex $v$ of $G$ such that $G'$ is connected and unicyclic. In this section, we characterize vertex $v$ of $G$ such that $G'$ is connected and bicyclic.

Theorem 2.1. Let $v \in V(G)$ be a non-cutvertex of a graph $G$ of order $p \geq 5$. Then $G'$ is connected and bicyclic if and only if either of the following holds:

a. $G = 2K_2U(p-4)K_1$, and $v$ is one of the $K_1$'s. \\
b. $G$ is connected, $G-v$ is unicyclic, $d_0(v) = |V(G)|-3$ and either the two non-adjacent vertices of $v$ do not lie on the cycle of $G-v$ or one of the two non-adjacent vertices of $v$ lie on the cycle and no vertex of the cycle lie on the path connecting the two non-adjacent vertices. \\
c. $G$ is connected, $G-v$ is bicyclic and $d_0(v) = |V(G)|-2$. \\
d. $G = DU[p-\lfloor V(D)\rfloor]K_1$, $G-v$ is bicyclic and $d_0(v) = |V(D)|-2$. \\
e. $G = DU[p-\lfloor V(D)\rfloor]K_1$, $G-v$ is unicyclic, $d_0(v) = |V(D)|-3$ and either the two non-adjacent vertices do not lie on the cycle of $G-v$ or one of the non-adjacent vertices lie on the cycle and no vertex of the cycle lie on the path connecting the two non-adjacent vertices. \\
f. $G = DU_2K_2U(p-4)\lfloor V(D)\rfloor)K_1$, $G-v$ is acyclic and $d_0(v) = |V(D)|-2$. \\
g. $G = DU_KU(p-2)\lfloor V(D)\rfloor)K_1$, $G-v$ is acyclic and $d_0(v) = |V(D)|-3$, where $D \neq K_1, K_2$ is a component of $G$ containing $v$.

Proof: Let $G'$ be connected and bicyclic. Then $G-v$ is either acyclic or unicyclic or bicyclic. Since $G'$ is connected using Theorem 1.3, $d_0(v) = 0$ or $d_0(v) < |V(B)|-1$ for every branch $B$ at $v$ in $G$.

If $d_0(v) = 0$, then $G$ is disconnected. Since $G'$ is bicyclic, we get $G = 2K_2U(p-4)K_1$, where $v$ is one of the $K_1$'s. Hence (a) is proved.

Now consider the case $d_0(v) \neq 0$ and $d_0(v) < |V(B)|-1$. That is, when $d_0(v) \leq |V(B)|-k, k = 2, 3, 4$ or $> 4$. Since $v$ is not a cutvertex of $G$, $B$ is either $G$ or a component of $G$ containing $v$ according as $G$ is connected or disconnected. We consider the following 6 cases with respect to $G$ and $G-v$.

Case 1: $G$ is connected and $G-v$ is acyclic.

Here $B = G$. Since $v$ is not a cutvertex of $G$, $G-v$ is connected. If $d_0(v) = |V(G)|-2$, then $G'$ is a tree, using Theorem 1.4. If $d_0(v) = |V(G)|-3$, then there exist exactly two vertices, say $u$ and $w$, in $G$ such that they are non-adjacent to $v$. Since $G-v$ is connected and acyclic, there exists a unique $u-w$ path in $G-v$ and hence in $G'$ also. Hence the edge $vu$, the path $u-w$ and the edge $wv$ form a unique cycle in $G'$, which is a contradiction. If $d_0(v) = |V(G)|-4$, then there exists exactly three vertices, say $u_1$, $u_2$ and $u_3$, in $G$ such that they are non-adjacent to $v$. Since $G-v$ is connected and acyclic, there exists unique $u_1-u_2$, $u_2-u_3$ and $u_1-u_3$ paths in $G-v$ and hence in $G'$ also. Now the edge $vu_1$, the path $u_1-u_2$ and the edge $wv$ form a cycle $C_1$, the edge $vu_2$, the path $u_2-u_3$ and the edge $u_3w$ form another cycle $C_2$ and the edge $vu_1$, the path $u_1-u_3$ and the edge $u_3w$ forms third cycle $C_3$ in $G'$, which is a contradiction. If $d_0(v) < |V(G)|-4$, then there exist at least 4 vertices which are non-adjacent to $v$ in $G$. Then by a similar argument given above, $G'$ has at least three cycles, which is a contradiction. Hence, in this case no graph $G$ exists.

Case 2: $G$ is connected and $G-v$ is unicyclic.

Let $C$ be the unique cycle in $G-v$. Then $C$ is also a cycle of $G'$ not containing $v$. Since $G'$ is connected, using Theorem 1.3, we have $d_0(v) \leq |V(G)|-2$. If $d_0(v) = |V(G)|-2$, then there exists exactly one vertex say $u$, which is non-adjacent to $v$ in $G$. Thus in $G'$, $v$ is adjacent only to $u$. Therefore in $G'$, $C$ is the unique cycle, which is a contradiction. If $d_0(v) < |V(G)|-3$, then $v$ is non-adjacent to at least three vertices say $u_1$, $u_2$ and $u_3$. Since $G-v$ is connected, there exist $u_1-u_2$, $u_2-u_3$ and $u_1-u_3$ paths in $G-v$ and hence in $G'$ also. Now the paths together with the edges $u_1v$, $u_2v$ and $u_3v$ form at least three cycles in $G'$, which is a contradiction. If $d_0(v) = |V(G)|-3$, then $v$ is non-adjacent to exactly two vertices, say $u$ and $w$. We consider the following three subcases.

Subcase 2.1: $u$ and $w$ do not lie on the cycle $C$.

Since $G-v$ is connected, there exists a $u-w$ path in $G-v$ and hence in $G'$ also. Now the edge $vu$, the path $u-w$ and the edge $wv$ form another cycle $C_1$ in $G'$, which contains the vertex $v$. Thus $G'$ is bicyclic.

Subcase 2.2: $u$ and $w$ lie on the cycle $C$.

Then there exists exactly two $u-w$ paths, say $P_1$ and $P_2$ in $G$ and hence, in $G'$ also. Now the edge $vu$, the path $P_1$ and the edge $wv$ form a cycle say $C_2$ in $G'$. Also the edge $vu$, the path $P_2$ and the edge $wv$ form another cycle say $C_3$ in $G'$. Thus $G'$ has three cycles, which is a contradiction.

Subcase 2.3: Either $u$ or $w$ lies on the cycle $C$.

Let us assume that $u$ lies on $C$. Since $G-v$ is connected, there exists either one $u-w$ path or two $u-w$ paths and hence, there arises two subcases.

Subcase 2.3.a: One $u-w$ path in $G-v$.

Then the $u-w$ path contains no vertex of the cycle $C$. Now the edge $vu$, the $u-w$ path and the edge $wv$ form a cycle different from $C$. Hence $G'$ is bicyclic.

Subcase 2.3.b: Two $u-w$ paths in $G-v$.
Let $P_3$ and $P_4$ be the two u-w paths in $G-v$. Then $vu$, $P_3$ and $vw$ form a cycle and $vu$, $P_4$ and $vw$ form another cycle in $G'$. Hence $G'$ has at least 3 cycles, which is a contradiction.

Hence, either the non-adjacent vertices $u$ and $w$ do not lie on the cycle $C$ or one of the non-adjacent vertices $u$ and $w$ of $v$ lies on $C$ and the u-w path does not contain any vertex of $C$. Also $d_G(v) = |V(G)|-3$. Thus (b) is proved.

**Case 3:** $G$ is connected and $G-v$ is bicyclic.

Let $C_1$ and $C_2$ be the two cycles in $G-v$. Then $v$ does not lie on both $C_1$ and $C_2$. Also $C_1$ and $C_2$ are the cycles in $G'$. Since $G'$ is connected, using Theorem 1.3, $d_G(v) \leq |V(G')|-2$. If $d_G(v) = |V(G')|-2$, then there exists exactly one vertex, say $u$, which is non-adjacent to $v$ in $G$. Hence $G' = G-v+uv$, and therefore $G'$ is bicyclic. If $d_G(v) < |V(G')|-2$, then there exists at least two vertices, say $u_1$ and $u_2$, which are non-adjacent to $v$. Since $G-v$ is connected, there exists a $u_1-u_2$ path in $G-v$ and hence in $G'$ also. Now the edge $vu_1$, the path $u_1-u_2$ and the edge $u_2-v$ together form another cycle which contains $v$, which is a contradiction. Thus the only possibility is $d_G(v) = |V(G')|-2$. Hence (c) is proved.

**Case 4:** $G$ is disconnected and $G-v$ is bicyclic.

Clearly $v$ is in a non-trivial component, say $D$, of $G$, since otherwise $G'$ is not bicyclic. Also the other components of $G$ are trivial graphs. This implies that $G = DU(p-|V(D)|)K_1$. Clearly, $G' = D'U(p-|V(D)|)(K_1+v)$. Since $G'$ is connected and bicyclic, $D'$ is also connected and bicyclic. Apply case-3 to $D$, we get $d_D(v) = |V(D')|-2$. Thus $G = DU(p-|V(D)|)K_1$ and $d_G(v) = |V(D')|-2$ and hence (d) is proved.

**Case 5:** $G$ is disconnected and $G-v$ is unicyclic.

Clearly $v$ is in a non-trivial component, say $D$, of $G$, since otherwise $G'$ is not bicyclic. Also the other components of $G$ are trivial graphs. This implies that $G = DU(p-|V(D)|)K_1$. Clearly, $G' = D'U(p-|V(D)|)(K_1+v)$. Since $G'$ is connected and bicyclic, $D'$ is also connected and bicyclic. Apply case-2 to $D$, we get $d_D(v) = |V(D')|-3$ and either the two non-adjacent vertices of $v$ do not lie on the cycle of $D-v$ or one of the two non-adjacent vertices lie on the cycle and no vertex of the cycle lie on the path connecting the two non-adjacent vertices. Thus $G = DU(p-|V(D)|)K_1$ and $d_G(v) = |V(D')|-3$ and either the two non-adjacent vertices of $v$ do not lie on the cycle of $G-v$ or one of the two non-adjacent vertices lie on the cycle and no vertex of the cycle lie on the path connecting the two non-adjacent vertices. Hence (e) is proved.

**Case 6:** $G$ is disconnected and $G-v$ is acyclic.

Here $v$ is in a non-trivial component, say $D$, of $G$. Since $v$ is not a cutvertex of $G$, $D$ is the only branch at $v$ in $G$. Since $G'$ is connected, $d_G(v) = d_D(v) \leq |V(D')|-2$, using Theorem 1.3. If $d_G(v) \leq |V(D')|-4$, then $D'$ has more than two cycles by a similar argument as in case-1. This implies that $d_G(v) = |V(D')|-2$ or $d_G(v) = |V(D')|-3$.

When $d_G(v) = |V(D')|-2$, $d_D(v) = |V(D')|-2$. Using Theorem 1.4, $D'$ is a tree. Since $G$ is disconnected, $G'$ has $k(G)$ branches at $v$ and the branch $D'$ is a tree where $k(G)$ is the number of components of $G$. Since $G'$ is connected, bicyclic and $G-v$ is acyclic, there exists exactly two branches, say $B_1$ and $B_2$, at $v$ in $G'$ are unicyclic, and the other $k(G)$-3 branches are trees. Let $D_1$ be the component of $G$ such that $B_1 = D_1+v$. Then $D_1 = K_2$, otherwise $B_1$ is not unicyclic. Using similar argument to $B_2$, we have $D_2 = K_2$. Also the other $k(G)$-3 components of $G$ are trivial graphs. This implies that $G = DU2K_1U(p-4|V(D)|)K_1$ and hence (f) is proved.

When $d_G(v) = |V(D')|-3$, $d_D(v) = |V(D')|-3$. As in case-1, we can prove that $D'$ is unicyclic. Since $G'$ is connected, bicyclic and since $D'$ is a unicyclic branch at $v$, exactly one more branch, say $B_3$ at $v$ in $G'$ is cyclic. Let $D_1$ be an acyclic component of $G$ such that $B_1 = D_1+v$. Then $D_1 = K_2$. If $G$ has a non-trivial component other than $D$, then $G'$ is not bicyclic, which is a contradiction. Hence, all the other components of $G$ are trivial graphs. Thus $G = DU2K_1U(p-2|V(D)|)K_1$ and hence (g) is proved.

Conversely, let either (a), (b), (c), (d), (e), (f) or (g) hold. From (a) to (g), we see that either $d_G(v) = 0$ or $d_G(v) < |V(B)|-1$ for the branch $B$ at $v$ in $G$. Using Theorem 1.3, $G'$ is connected. It is noted that for the branch $B(G)$ or $D$ at $v$ in $G$, if $d_G(v) = |V(B')|-2$, then $B'$ is a tree or bicyclic branch at $v$ in $G'$ according as $G-v$ is acyclic or bicyclic and if $d_G(v) = |V(B')|-3$, then $B'$ is a unicyclic or bicyclic branch at $v$ in $G'$ according to $G-v$ is acyclic or unicyclic. Then each case implies that $G^v$ is bicyclic. Hence the theorem is proved.

**Theorem 2.2.** Let $v \in V(G)$ be a cutvertex of a graph $G$ of order $p \geq 5$. Then $G^v$ is connected and bicyclic if and only if either of the following holds:

a. $G$ is connected, $G-v$ is acyclic, $d_G(v) \in \{|V(B)|-2, |V(B)|-3\}$ for any branch $B$ at $v$ in $G$ and $d_G(v) = |V(B)|-3$ exactly for two $B$'s.

b. $G$ is connected, $G-v$ is unicyclic, $d_G(v) \in \{|V(B)|-2, |V(B)|-3\}$ for any branch $B$ at $v$ in $G$ and $d_G(v) = |V(B)|-3$ only for one $B = B^*$ and $B^*-v$ is unicyclic.

c. $G$ is connected, $G-v$ is bicyclic, $d_G(v) = |V(B)|-2$ for any branch $B$ at $v$ in $G$ and $B-v$ is bicyclic for only one $B$.

d. $G = DU(p-|V(D)|)K_1$, $G-v$ is acyclic, $d_G(v) \in \{|V(B)|-2, |V(B)|-3\}$ for any branch $B$ at $v$ in $G$ and $d_G(v) = |V(B)|-3$ for exactly two $B$'s.

e. $G = DU2K_1U(p-4|V(D)|)K_1$, $G-v$ is acyclic and $d_G(v) = |V(B)|-2$ for any branch $B$ at $v$ in $G$.

f. $G = DU2K_1U(p-2|V(D)|)K_1$, $G-v$ is bicyclic, $d_G(v) \in \{|V(B)|-2, |V(B)|-3\}$ for any branch $B$ at $v$ in $G$ and $d_G(v) = |V(B)|-3$ only for one $B$.

g. $G = DU(p-|V(D)|)K_1$, $G-v$ is unicyclic, $d_G(v) \in \{|V(B)|-2, |V(B)|-3\}$ for any branch $B$ at $v$ in $G$ and $d_G(v) = |V(B)|-3$ only for one $B = B^*$ and $B^*-v$ is unicyclic.

h. $G = DU(p-|V(D)|)K_1$, $G-v$ is bicyclic, $d_G(v) = |V(B)|-2$, for any branch $B$ at $v$ in $G$ and $B-v$ is bicyclic for one $B$, where $D \neq K_2$, $K_2$ is a component of $G$ containing $v$.

**Proof:** Let $G^v$ be connected and bicyclic. Using Theorem 1.3, $d_G(v) = 0$ or $d_G(v) \leq |V(B')|-2$ for every branch $B$ at $v$ in $G$. Since $v$ is a cutvertex of $G$, $d_G(v) \neq 0$. Therefore the only possibility is $d_G(v) \leq |N(B')|-2$ for every branch $B$ at $v$ in $G$. Since $G'$ is bicyclic, $G-v$ is either bicyclic or unicyclic or acyclic. Here $G$ may be either connected or disconnected and correspondingly we consider the following cases:

**Case 1:** $G$ is connected.
Let $k$ be the number of branches at $v$ in $G$. Using Theorem 1.2, there are only $k$ branches at $v$ in $G'$, since $G'$ is connected. Since $G'$ is bicyclic, there are two possibilities for the bicycle. Either exactly two branches at $v$ in $G'$ are unicyclic, or only one branch at $v$ in $G'$ is bicyclic. All other branches at $v$ in $G'$ are trees. Let $B$ be any branch at $v$ in $G$ such that $B^*$ is a tree in $G'$. Using Theorem 1.4, $d_B(v) = |V(B)|-2$.

**Subcase 1.1:** Exactly two branches at $v$ in $G'$ are unicyclic, say $B_1$ and $B_2$.

Let $B_1^* = B_1^-$ and $B_2^* = B_2^-$. Then both $B_1^*$ and $B_2^*$ are branches at $v$ in $G$. If $d_{B_1^*}(v) < |V(B_1^*)|-3$, $i = 1, 2$, then there exists at least 3 vertices, say $u_{i1}, u_{i2}, u_{i3}$ which are non-adjacent to $v$ in $B_i^*$. Since $B_i^*$ is connected, there exists $u_{i1} - u_{i2} - u_{i3}$ and $u_{i1} - u_{i2}$ paths in $B_i^*$. Now these paths and the edges $u_{i1}v, u_{i2}v$ and $u_{i3}v$ together form at least three cycles in $B_i$, $i = 1, 2$, which is a contradiction. Thus $d_{B_i^*}(v) = |V(B_i^*)|-3, i = 1, 2$. Also $G$-$v$ is acyclic. Thus (a) is proved.

**Subcase 1.2:** One branch, say $B_3$ at $v$ in $G'$ is bicyclic.

Here there are two possibilities. Let $B^* = B_3^*$. Then $B^*$ is a branch at $v$ in $G$. Either $v$ lies on a cycle of $B_3$. or $B_3$ is a tree containing $v$.

If $v$ lies on the cycle of $B_3$, then $B_3$-$v$ is unicyclic and hence $G$-$v$ is unicyclic. Also $d_{B_3^*}(v) = |V(B_3^*)|-3$. Thus (b) is proved.

If $v$ does not lie on the cycles of $B_3$, then $B_3$-$v$ is bicyclic and hence $G$-$v$ is bicyclic. Also $d_{B_3^*}(v) = |V(B_3^*)|-2$. Thus (c) is proved.

**Case 2:** $G$ is disconnected and $G$-$v$ is acyclic.

Let $D$ be a component of $G$ containing $v$. Since $v$ is a cutvertex, $D$ is neither $K_1$ nor $K_2$. Since $G$-$v$ is acyclic and $G'$ is bicyclic, there must be exactly two unicyclic branches at $v$ in $G'$ each contains $v$. Let them be $B_1$ and $B_2$. Here we consider three subcases $B_1 \neq K_3$, $B_2 \neq K_3$, $i = 1, 2$ and either $B_1 = K_3$ or $B_2 = K_3$.

**Subcase 2.1:** $B_i \neq K_3$, $i = 1, 2$.

Here $|B_i| \geq 4$, $i = 1, 2$. Let $F_1$ be a component of $G$ not containing $v$ such that $F_1 + v = B_i$, $i = 1, 2$. Then $|V(F_1)| \geq 3$, which implies that $B_i$, $i = 1, 2$ are not unicyclic. This is a contradiction to our assumption that $B_i$, $i = 1, 2$ are unicyclic.

Hence $B_i$ is obtained from a branch, say $B_i^*$ at $v$ in $G$ and $B_i = B_i^*$, $i = 1, 2$. Now $B_i^*$ is connected and $B_i^*$-$v$ is acyclic, $i = 1, 2$. Using Theorem 1.7(b) to $B_i^*$ we get $d_{B_i^*}(v) = |V(B_i^*)|-3, i = 1, 2$. Let $B \neq B_i^*$, $i = 1, 2$ be a branch at $v$ in $G$. Clearly $d_B(v) = |V(B)|-2$ since otherwise the branch $B^*$ at $v$ in $G'$ has a cycle. Also each component of $G$ other than $D$ is a trivial graph. This implies that $G = DU_{K_2}U(p-4-[V(D)])|K_1$. Thus (d) is proved.

**Subcase 2.2:** $B_1 = K_3$, $i = 1, 2$.

Using Theorem 1.5, $B_1$-$v = K_2$, $i = 1, 2$ is a component of $G$. Since $G'$ is bicyclic, each component other than $D$ and $K_3$ is $K_1$. This implies that $G = DU_{K_2}U(p-4-[V(D)])|K_1$. Let $B$ be any branch at $v$ in $G$. Since $G'$ is connected, $B^*$ is a branch at $v$ in $G'$. Since the unicyclic branches at $v$ in $G'$ corresponds to the component $K_2$ of $G$, the branch $B^*$ is a tree at $v$ in $G'$. Using Theorem 1.4, we get $d_B(v) = |V(B)|-2$. Thus (e) is proved.

**Subcase 2.3:** Either $B_1$ or $B_2$ is $K_3$.

Let $B_1 = K_3$ and $B_2 \neq K_3$. Using Theorem 1.5, $B_1$-$v = K_2$ is a component of $G$. Since $G'$ is bicyclic, each component other than $D$ and $K_2$ is $K_1$. This implies that $G = DU_{K_2}U(p-4-[V(D)])|K_1$. Since $B_2 \neq K_3$, $|B_2| \geq 4$. Let $F$ be a component of $G$ not containing $v$ such that $F+v= B_2$. Then $|V(F)| \geq 3$, which implies that $B_2$ is not unicyclic. This is a contradiction to our assumption that $B_2$ is unicyclic. Hence $B_2$ is obtained from a branch, say $B_2^*$ at $v$ in $G$ and $B_2 = B_2^*$. Now $B_2^*$ is connected and $B_2^*$-$v$ is acyclic. Using Theorem 1.7(b) to $B_2^*$, we get $d_{B_2^*}(v) = |V(B_2^*)|-3$. Let $B \neq B_2^*$ be a branch at $v$ in $G$. Since $G'$ is connected, $B^*$ is a branch at $v$ in $G'$. Since the unicyclic branch $B_1$ at $v$ in $G'$ corresponds to the component $K_2$ of $G$, the unicyclic branch $B_2$ at $v$ in $G'$ corresponds to a branch at $v$ of the component $D$ of $G$ and $d_{B_2^*}(v) = |V(B_2^*)|-3$, the branch $B^*$ is a tree at $v$ in $G'$. Using Theorem 1.4, we get $d_B(v) = |V(B)|-2$. Thus (f) is proved.

**Case 3:** $G$ is disconnected and $G$-$v$ is unicyclic.

Clearly, $v$ is in a non-trivial component, say $D$ of $G$, since otherwise $G'$ is not bicyclic. Also the other components of $G$ are trivial graphs. This implies that $G = DU(p-|V(D)|)|K_1$ and hence $G' = D'U(p-|V(D)|)|K_1+v$. Since $G'$ is bicyclic, $D'$ is bicyclic. Since $D$ is connected and $D$-$v$ is unicyclic, using (b) to $D$, we get $d_B(v) \in \{ |(V(B)|-2, |(V(B)|-3 \}$ for any branch $B$ at $v$ in $G(D)$ and $d_B(v) = |V(B)|-3$ for only one $B = B^*$. Clearly $B^*$-$v$ is unicyclic. Thus (g) is proved.

**Case 4:** $G$ is disconnected and $G$-$v$ is bicyclic.

Clearly, $v$ is in a non-trivial component, say $D$ of $G$, since otherwise $G'$ is not bicyclic. Also the other components of $G$ are trivial graphs. This implies that $G = DU(p-|V(D)|)|K_1$ and hence $G' = D'U(p-|V(D)|)|K_1+v$. Since $G'$ is bicyclic, $D'$ is bicyclic. Since $D$ is connected and $D$-$v$ is bicyclic. Using (c) to $D$, we get $d_B(v) = |V(B)|-2$ for any branch $B$ at $v$ in $G$ and $B$-$v$ is bicyclic for only one $B$. Thus (h) is proved.

Conversely, let either (a), (b), (c), (d), (e), (f), (g) or (h) hold. From (a) to (h), we see that $d_B(v) \leq |V(B)|-2$ for any branch $B$ at $v$ in $G$ and hence using Theorem 1.3, $G'$ is connected. Clearly each case implies that $G'$ is bicyclic. Hence the theorem is proved.

**Note 2.3.[1]** Consider a cycle $C_c = (v_1, v_2, \ldots, v_c)$ (clockwise). For our convenience we denote it by $C_{(v1)}$. Identifying an end vertex of paths $P_m$ at $v_1$, $P_s$ at $v_s$, then $C_{(v1)}$ is denoted by $C_{(v1)}(0, \ldots, P_m, 0, \ldots, P_s, 0, \ldots, 0)$. Identifying an end vertex of paths $P_m$ and $P_s$ at the vertex $v_1$, then $C_{(v1)}$ is denoted by $C_{(v1)}(0, \ldots, P_m, P_s, 0, \ldots, 0, 0, \ldots, 0)$.

The graphs $C_{(v1)}(0, 0, P_2, P_3)$, $C_{(v1)}(0, 2P_2, P_3, 0, 0)$ and $C_{(v1)}(0, 2P_2, 2P_3, 0, P_2, P_3)$ are given in Figure 2.1.

Note 2.4.[5] Let $v$ be a cutvertex of a connected graph $G$. Let $B_1, B_2, \ldots, B_n$ be the branches with $n_1, n_2, \ldots, n_n$ number of copies at $v$ in $G$, respectively. In this case, we denote the graph $G$ by $G(v; n_1B_1, n_2B_2, \ldots, n_nB_n)$.

III. CHARACTERIZING CONNECTED BICYCLIC GRAPHS WITH A SELF VERTEX SWITCHING

Theorem 3.1. Let $G$ be a connected bicyclic graph of odd order $p = 2n+1$. Then $G$ has a self vertex switching $v$ if and only if $G$ is either of the following: $G(v; (n-6)P_3, 2C_4, 2K_{1,3})$ or $G(v; (n-6)P_3, 2C_{4w}(P_2, 0, 0), 2P_3)$ or $G(v; (n-6)P_3, C_4, K_{1,3}, C_{3w}(P_2, 0, 0), P_3)$ or $G(v; (n-4)P_3, C_5, C_{3w}(P_2, 0, 0))$ or $G(v; (n-4)P_3, 2C_{4w}(P_2, 0, 0, 0))$ and $v$ is adjacent to $w$ or $G(v; (n-4)P_3, C_{4w}(P_2, 0, 0, 0), C_{3w}(2P_2, 0, 0))$ and $v$ is non-adjacent to $w$ in $C_{3w}(2P_2, 0, 0)$ or $G(v; (n-4)P_3, 2C_{4w}(P_3, 0, 0))$ and $v$ is non-adjacent to the end vertex or $G(v; (n-5)P_3, C_4, K_{1,3}, C_{3w}(P_2, 0, 0, 0))$ and $v$ is adjacent to $w$ or $G(v; (n-5)P_3, C_4, K_{1,3}, C_{3w}(P_2, 0, 0, 0))$ and $v$ is adjacent to $w$ but non-adjacent to the end vertices or $G(v; (n-5)P_3, C_{3w}(P_2, 0, 0, 0), P_4, C_{4w}(P_2, 0, 0, 0))$ and $v$ is adjacent to $w$ or $G(v; (n-5)P_3, C_{3w}(P_2, 0, 0, 0), P_4, C_{4w}(P_2, 0, 0, 0))$ and $v$ is adjacent to $w$ but non-adjacent to the end vertices, where $w$ is a vertex of degree 3 and for every branch $B$ at $v$ in $G$, $d_B(v) = 1$ or 2 according as $B$ is a tree or unicyclic branch at $v$ in $G$.

Proof: Let $v$ be a self vertex switching of a connected bicyclic graph $G$. Then $G \cong G'$ and hence $G$ has at least 5 vertices. Using Theorem 1.1, $d_G(v) = (p-1)/2 = n$. Here $v$ may be either a cutvertex or not and correspondingly we consider the following two cases.

Case 1: $v$ is not a cutvertex of $G$.

Using Theorem 2.1, either $G-v$ is unicyclic, $d_G(v) = [V(G)]-3$ and two non-adjacent vertices of $v$ do not lie on the cycle of $G-v$ or $G-v$ is uniclic, $d_G(v) = [V(G)]-3$, one of the two non-adjacent vertices of $v$ lie on the cycle and no vertex of the cycle lie on the path connecting the two non-adjacent vertices or $G-v$ is bicylic and $d_G(v) = [V(G)]-2$.

Subcase 1.1: $G-v$ is unicyclic, $d_G(v) = [V(G)]-3$ and the two non-adjacent vertices of $v$ do not lie on the cycle of $G-v$.

Since $G$ is bicyclic and $G-v$ is uniclic, $v$ must lie on any one of the cycle, say $C$. Since $v$ is not a cut vertex of $G$, $d_G(v) = 2 = (p-1)/2$ and hence $p = 5$. The only bicyclic graph on 5 vertices is given in Figure 3.1. This graph has no self vertex switching and so no such graph $G$ exists.

Figure 3.1

Subcase 1.2: $G-v$ is unicyclic, $d_G(v) = [V(G)]-3$, one of the two non-adjacent vertices of $v$ lie on the cycle and no vertex of the cycle lie on the path connecting the two non-adjacent vertices. By a similar argument given in subcase 1.1, no such graph $G$ exists.

Subcase 1.3: $G-v$ is bicyclic and $d_G(v) = [V(G)]-2$.

Since $G$ and $G-v$ are bicyclic, $v$ does not lie on both cycles of $G$. Since $G-v$ is connected and $v$ is not a cutvertex of $G$, $v$ is an end vertex of $G$ and hence $p = 3$. But there is no bicyclic graph on 3 vertices. Hence there is no connected bicyclic graph $G$ such that $G-v$ is bicyclic and $d_G(v) = [V(G)]-2$.

Case 2: $v$ is a cutvertex of $G$.

Using Theorem 2.2, either $G-v$ is acyclic, $d_B(v) \in \{[V(B)]-2, [V(B)]-3\}$ for any branch $B$ at $v$ in $G$ and $d_B(v) = [V(B)]-3$ exactly for two $B$’s or $G-v$ is uniclic, $d_B(v) \in \{[V(B)]-2, [V(B)]-3\}$ for any branch $B$ at $v$ in $G$, $d_B(v) = [V(B)]-3$ only for one $B = B^*$ and $B^{*\prime}$ is uniclic or $G-v$ is bicyclic, $d_B(v) = [V(B)]-2$ for any branch $B$ at $v$ in $G$ and $B-v$ is bicyclic for only one $B$ and hence we have the following three subcases.

Subcase 2.1: $G-v$ is acyclic, $d_B(v) \in \{[V(B)]-2, [V(B)]-3\}$ for any branch $B$ at $v$ in $G$ and $d_B(v) = [V(B)]-3$ exactly for two $B$’s.

Since $G-v$ is acyclic, $v$ lies on both the cycles. Let $B_1$ and $B_2$ be the two cyclic branches at $v$ in $G$ and $d_{B_1}(v) = d_{B_2}(v) = 2$. Since $G$ is bicyclic, any branch $B \neq B_1$ and $B_2$ at $v$ in $G$ is a tree. Using Theorem 1.6, $[V(B)] \geq 3$. Since $d_{B_1}(v) = n$, there are only $n-2$ branches at $v$ in $G$. If $[V(B)] > 6$, $i = 1,2$, then $p \geq ([n-2)-2]+6+6+(n-3) = (n-4)+3+12+(n-3) = 3n-12+12n+3 = 2n+3 = p+2 > p$, which is a contradiction. If $[V(B)] = 3$, then $B_1-v$ and $B_2-v$ are the components of $G'$ and hence $G$ is disconnected, which is a contradiction. This implies that $[V(B)] = 4$ or 5, $i = 1,2$.

Subcase 2.1.a: $[V(B)] = 4, i = 1,2$.

The only uniclic graphs on 4 vertices are $C_4$ and $C_{3w}(P_2, 0, 0)$ and they are given in Figure 3.2. Hence $B_1$ is either $C_4$ or $C_{3w}(P_2, 0, 0)$.

Figure 3.2

The switching of a vertex $v$ of degree 2 in $C_4$ and $C_{3w}(P_2, 0, 0)$ are $K_{1,3}$ and $P_4$, respectively. Clearly $C_4$ and $K_{1,3}$ and $P_4$. 

Figure 2.3

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C_{3w}(P_2, 0, 0) and P_4 are complementary switching branches at v. If B is a branch at v in G such that B ≠ B_b, B ≠ B_w and of order at least 4, then p = |V(B)| + |V(B_v)| + |V(B_{1,b})| + |V(B_{2,b})| + (number of vertices in the remaining (n-7) branches at v in G) - (n-3) ≥ 4+4+4+4+4+(n-3) = 20+3n-21+n-3 = 2n+2 > p, which is a contradiction and hence B is of order 3. Since B is a tree, B = P_3. This implies that G is either G(v; (n-6)P_3, 2C_4, 2K_{1,3}) or G(v; (n-6)P_3, 2C_4\{P_2, 0, 0\}, 2P_2) or G(v; (n-6)P_3, C_4, K_{1,3}, C_{3w}(P_2, 0, 0, P_4)) where w is a vertex of degree 3 adjacent to v in G, d_{3w}(v) = 1 or 2 according as B is not a cyclic branch or a cyclic branch at v in G.

Subcase 2.1.b: |V(B)|= 5, i.e., 1, 2.

There are only five unicyclic graphs on 5 vertices, which are given in Figure 3.3.

![Figure 3.3](image)

For any vertex v in C_3, the switching of C_4 is C_{3w}(0, P_2, P_2). Hence C_3 and C_{3w}(0, P_2, P_2) are complementary switching branches at v. The switching of a vertex v(u) of degree 2 which is adjacent to the vertex w of degree 3 in C_{3w}(P_2, 0, 0, 0) is itself. Hence C_{3w}(P_2, 0, 0, 0) is a self switching branch at v(u). The switching of a vertex v(u) of degree 2 and is non-adjacent to the vertex w of degree 3 in C_{4w}(P_2, 0, 0, 0) is C_{3w}(2P_2, 0, 0). Thus C_{4w}(P_2, 0, 0, 0) and C_{3w}(2P_2, 0, 0) are complementary switching branches at v(u). For any vertex v of degree 2 which is non-adjacent to the end vertex in C_{3w}(P_2, 0, 0), the switching is itself. For the vertex of degree 2 which is adjacent to the end vertex in C_{3w}(P_2, 0, 0), the switching is a disconnected graph. Hence C_{3w}(P_3, 0, 0) is a self switching branch at v(u). Clearly the other branches are P_1's. This implies that G is either G(v; (n-4)P_3, C_4, C_{3w}(0, P_2, P_2)) or G(v; (n-4)P_3, 2C_4\{P_2, 0, 0\}, 0) or G(v; (n-4)P_3, C_{4w}(2P_2, 0, 0, 0)) or G(v; (n-4)P_3, 2C_{4w}(P_2, 0, 0, 0)) where w is a vertex of degree 3 in G, d_{3w}(v) = 1 or 2 according as B is not a cyclic branch or a cyclic branch at v in G and v is non-adjacent to any end vertex.

Subcase 2.1.c: |V(B_v)|= 4 and |V(B_{2,b})|= 5.

By using subcase 2.1.a, we see that B_1 is either C_4 or C_{3w}(P_2, 0, 0) and using subcase 2.1.b, we see that B_2 is either C_3 or C_{4w}(P_2, 0, 0, 0) or C_{3w}(P_2, 0, 0, 0) or C_{3w}(P_2, 0, 0) or C_{3w}(0, P_2, P_2). B_1 and B_2\' are complementary switching branches at v in G. For any vertex which is non-adjacent to an end vertex of degree 2 in B_2, B_2\' is unicyclic. Since G is bicyclic and G \cong G', B_2 must be either C_{3w}(P_2, 0, 0, 0) or C_{3w}(P_2, 0, 0). Clearly the other branches are P_1's. This implies that G is either G(v; (n-5)P_3, C_4, K_{1,3}, C_{4w}(P_2, 0, 0)) and v is adjacent to w or G(v; (n-5)P_3, C_4, K_{1,3}, C_{4w}(P_2, 0, 0)) and v is adjacent to w but non-adjacent to the end vertices or G(v; (n-5)P_3, C_{3w}(P_2, 0, 0, 0)) and v is adjacent to w or G(v; (n-5)P_3, C_{3w}(P_2, 0, 0, 0)) and v is adjacent to w but non-adjacent to the end vertices where w is a vertex of degree 3 and d_{3w}(v) = 1 or 2 according as B is not a cyclic branch or a cyclic branch at v in G.

Subcase 2.2: G-v is unicyclic, d_{3w}(v) \in \{ |V(B)|-2, |V(B)|-3 \} for any branch B at v in G, d_{3w}(v) = |V(B)|-3 only for one B = B_1 and B_2\' is unicyclic.

Since G-v is unicyclic, exactly one B-v is unicyclic where B is a branch at v in G. If d_{3w}(v) = |V(B)|-3, then the branch B\' at v in G' has at least 3 cycles and hence G is not bicyclic. If d_{3w}(v) = |V(B)|-2, then the branches B\' at v in G' is unicyclic and hence G is not bicyclic. In this case there is no connected bicyclic graph exists.

Subcase 2.3: G-v is bicyclic, d_{3w}(v) = |V(B)|-2 for any branch B at v in G and B-v is bicyclic for only one B.

B-v is bicyclic and d_{3w}(v) = |V(B)|-2 implies that the branch B at v in G has at least 3 cycles and hence G is not bicyclic. Hence in this case there is no connected bicyclic graph G exists.

Thus from the above cases, we see that G is either G(v; (n-6)P_3, 2C_4, 2K_{1,3}) or G(v; (n-6)P_3, 2C_4\{P_2, 0, 0\}, 2P_2) or G(v; (n-6)P_3, C_4, K_{1,3}, C_{3w}(P_2, 0, 0, P_4)) or G(v; (n-4)P_3, 2C_4\{P_2, 0, 0\}) or G(v; (n-4)P_3, 2C_{4w}(P_2, 0, 0, 0)) and v is adjacent to w. G(v; (n-4)P_3, C_{4w}(2P_2, 0, 0, 0), 0)) and v is adjacent to w in C_{4w}(P_2, 0, 0, 0) but non-adjacent to w in C_{4w}(P_2, 0, 0, 0) or G(v; (n-4)P_3, 2C_{4w}(P_2, 0, 0, 0)) and v is adjacent to w or G(v; (n-5)P_3, C_{3w}(0, P_2, 0, 0)) and v is adjacent to w and non-adjacent to the end vertices where w is a vertex of degree 3 and d_{3w}(v) = 1 or 2 according as B is not a cyclic branch or a cyclic branch at v in G.

Conversely, let G be the graph given in the theorem. Then, clearly each case implies that v is a self vertex switching of G. This completes the proof.

Corollary 3.2 Let G be a connected bicyclic graph. Then s_{s}(G) = 0 or 1. s_{s}(G) = 1 if and only if G is either G(v; (n-6)P_3, 2C_4, 2K_{1,3}) and n \geq 6 or G(v; (n-6)P_3, 2C_{4w}(P_2, 0, 0), 2P_2) and n \geq 6 or G(v; (n-6)P_3, C_4, K_{1,3}, C_{3w}(P_2, 0, 0, P_4)) and n \geq 4 or G(v; (n-4)P_3, 2C_{4w}(P_2, 0, 0, 0)) and v is adjacent to w and n \geq 4 or G(v; (n-4)P_3, C_{4w}(2P_2, 0, 0, 0)) and v is adjacent to w in C_{4w}(2P_2, 0, 0) but non-adjacent to w in C_{4w}(P_2, 0, 0, 0) and n \geq 4 or G(v; (n-4)P_3, 2C_{4w}(P_2, 0, 0, 0)) and v is adjacent to w but non-adjacent to the end vertices with n \geq 4 or G(v; (n-5)P_3, C_{3w}(0, P_2, 0, 0)) and v is adjacent to w but non-adjacent to the end vertices and n \geq 5 or G(v; (n-5)P_3, C_{3w}(0, P_2, 0, 0)) and v is adjacent to w and n \geq 5 or G(v; (n-5)P_3, C_{3w}(0, P_2, 0, 0)) and v is adjacent to w but non-adjacent to the end vertices.

Example 3.3 The connected bicyclic graphs corresponding to n = 7 (p = 15) are given in Figure 3.4. In each graph, v is the self vertex switching.
REFERENCES