Perturbation of AC - Mixed Type Functional Equation

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Abstract— In this paper, we obtain the general solution and generalized Ulam - Hyers stability of a 2 variable AC mixed type functional equation by direct method.

\[ f(2x+y) - f(2x-y) = 4\left[f(x+y) - f(x-y)\right] - 6f(y) \]

I. INTRODUCTION

The study of stability problems for functional equations is related to a question of Ulam [19] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [6]. It was further generalized and excellent results obtained by number of authors [2, 4, 11, 14, 16].

Over the last six or seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involves only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 3, 7, 9, 10, 12, 13, 15, 17].

The general solution and Ulam stability of mixed type additive and cubic functional equation of the form

\[ 3f(x+y+z) + f(-x+y+z) + f(x-y+z) + f(x+y-z) + 4[f(x)+f(y)+f(z)] = 4[f(x+y)+f(x+z)+f(y+z)] \]

introduced by J.M. Rassias [12]. The stability of generalized mixed type functional equation of the form

\[ f(x+ky) + f(x-ky) = k^2[f(x+y) + f(x-y)] + 2(1-k^2)f(x) \]

for fixed integers \(k\) with \(k \neq 0, \pm 1\) in quasi-Banach spaces was investigated by M. Eshaghi Gordji and H. Khodaie [5].

In this paper, the authors introduced and investigated AC - mixed type functional equation

\[ f(2x+y) - f(2x-y) = 4[f(x+y) - f(x-y)] - 6f(y) \]

having solution

\[ f(x) = ax + bx^3 \]

In Section 2, we present the general solution of the functional equation (3). The generalized Ulam-Hyers stability using direct method for additive, cubic and mixed are discussed in Section 3, 4 and 5, respectively.

II. GENERAL SOLUTION

In this section, we present the solution of the functional equation (3). Through out this section let \(U\) and \(V\) be real vector spaces.

Lemma 2.1: If \(f: U \rightarrow V\) be a mapping satisfying (3) and let \(g: U \rightarrow V\) be a mapping given by

\[ g(x) = f(2x) - 8f(x) \]
for all $x \in U$ then
\[ g(2x) = 2g(x) \] (6)
for all $x \in U$ such that $g$ is additive.

*Proof.* Letting $(x, y)$ by $(0,0)$ in (3), we get
\[ f(0) = 0. \] (7)
Setting $(x, y)$ by $(0, y)$ in (3), we obtain
\[ f(-y) = -f(y) \] (8)
for all $y \in U$. Replacing $(x, y)$ by $(x, x)$ in (3), we arrive
\[ f(3x) = 4f(2x) - 5f(x) \] (9)
for all $x \in U$. Again replacing $(x, y)$ by $(x, 2x)$ in (3) and using (7),(8) and (9), we have
\[ f(4x) = 10f(2x) - 16f(x) \] (10)
for all $x \in U$. From (5), we establish
\[ g(2x) - 2g(x) = f(4x) - 10f(2x) + 16f(x) \] (11)
for all $x \in U$. Using (10) in (11), we desired our result.

*Lemma 2.2.* If $f : U \rightarrow V$ be a mapping satisfying (3) and let $h : U \rightarrow V$ be a mapping given by
\[ h(x) = f(2x) - 2f(x) \] (12)
for all $x \in U$ then
\[ h(2x) = 8h(x) \] (13)
for all $x \in U$ such that $h$ is cubic.

*Proof.* It follows from (12) that
\[ h(2x) - 8h(x) = f(4x) - 10f(2x) + 16f(x) \] (14)
for all $x \in U$. Using (10) in (14), we desired our result.

*Remark 2.3.* If $f : U \rightarrow V$ be a mapping satisfying (3) and let $g, h : U \rightarrow V$ be a mapping defined in (5) and (12) then
\[ f(x) = \frac{1}{6}(h(x) - g(x)) \] (15)
for all $x \in U$.

Hereafter through out this paper, let $U$ be a normed space and $V$ be a Banach space. Define a mapping $F : U \rightarrow V$ by
\[ F(x, y) = f(2x + y) - f(2x - y) - 4f(x + y) + 4f(x - y) + 6f(y) \]
for all $x, y \in U$.

**III. STABILITY RESULTS: ADDITIVE CASE**

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (3) for additive case.

*Theorem 3.1.* Let $j = \pm 1$. Let $f : U \rightarrow V$ be a mapping for which there exist a function $\alpha : U^2 \rightarrow [0, \infty)$ with the condition
\[
\lim_{n \to \infty} \frac{1}{2^n} \alpha(2^n x, 2^n y) = 0 \tag{16}
\]

such that the functional inequality
\[
\|F(x, y)\| \leq \alpha(x, y) \tag{17}
\]

for all \( x, y \in U \). Then there exists a unique additive mapping \( A : U \to V \) satisfying the functional equation (3) and
\[
\left\| f(2x) - 8f(x) - A(x) \right\| \leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \beta(2^k x) \tag{18}
\]

for all \( x \in U \), where \( \beta(2^k x) \) and \( A(x) \) are defined by
\[
\beta(2^k x) = 4\alpha(2^k x, 2^k x) + \alpha(2^k x, 2^{(k+1)} x) \tag{19}
\]

and
\[
A(x) = \lim_{n \to \infty} \frac{1}{2^n} (f(2^{(n+1)} x) - 8f(2^n x)) \tag{20}
\]

for all \( x \in U \), respectively.

Proof. Assume \( j = 1 \). Letting \((x, y)\) by \((x, x)\) in (17), we obtain
\[
\|f(3x) - 4f(2x) + 5f(x)\| \leq \alpha(x, x) \tag{21}
\]

for all \( x \in U \). Replacing \((x, y)\) by \((x, 2x)\) in (17), we get
\[
\|f(4x) - 4f(3x) + 6f(2x) - 4f(x)\| \leq \alpha(x, 2x) \tag{22}
\]

for all \( x \in U \). Now, from (21) and (22), we have
\[
\left\| f(4x) - 10f(2x) + 16f(x) \right\|
\leq 4\left\| f(3x) - 4f(2x) + 5f(x) \right\| + \left\| f(4x) - 4f(3x) + 6f(2x) - 4f(x) \right\|
\leq 4\alpha(x, x) + \alpha(x, 2x) \tag{23}
\]

for all \( x \in U \). From (23), we arrive
\[
\left\| f(4x) - 10f(2x) + 16f(x) \right\| \leq \beta(x) \tag{24}
\]

where
\[
\beta(x) = 4\alpha(x, x) + \alpha(x, 2x) \tag{25}
\]

for all \( x \in U \). It is easy to see from (24) that
\[
\left\| f(4x) - 8f(2x) - 2(f(2x) - 8f(x)) \right\| \leq \beta(x) \tag{26}
\]

for all \( x \in U \). Using (5) in (26), we obtain
\[
\left\| g(2x) - 2g(x) \right\| \leq \beta(x) \tag{27}
\]

for all \( x \in U \). From (27), we arrive
\[
\left\| \frac{g(2x)}{2} - g(x) \right\| \leq \frac{\beta(x)}{2} \tag{28}
\]

for all \( x \in U \). Now replacing \( x \) by \( 2x \) and dividing by \( 2 \) in (28), we get
for all \( x \in U \). From (28) and (29), we obtain
\[
\left\| \frac{g(2^2x)}{2^2} - g(x) \right\| \leq \left\| \frac{g(2x)}{2} - g(x) \right\| + \left\| \frac{g(2^2x)}{2^2} - \frac{g(2x)}{2} \right\|
\leq \frac{1}{2} \left[ \beta(x) + \frac{\beta(2x)}{2} \right]
\]
for all \( x \in U \). Proceeding further and using induction on a positive integer \( n \), we get
\[
\left\| \frac{g(2^nx)}{2^n} - g(x) \right\| \leq \frac{1}{2} \sum_{k=0}^{n-1} \beta(2^k x)
\leq \frac{1}{2} \sum_{k=0}^{\infty} \beta(2^k x) 2^k
\]
for all \( x \in U \). In order to prove the convergence of the sequence
\[
\left\{ \frac{g(2^n x)}{2^n} \right\},
\]
replacing \( x \) by \( 2^mx \) and dividing by \( 2^m \) in (31), for any \( m, n > 0 \), we deduce
\[
\left\| \frac{g(2^{n+m} x)}{2^{n+m}} - \frac{g(2^m x)}{2^m} \right\| = \frac{1}{2^m} \left\| \frac{g(2^n \cdot 2^m x)}{2^n} - g(2^m x) \right\|
\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\beta(2^{k+m} x)}{2^{k+m}}
\rightarrow 0 \text{ as } m \rightarrow \infty
\]
for all \( x \in U \). This shows that the sequence \( \left\{ \frac{g(2^n x)}{2^n} \right\} \) is Cauchy sequence. Since \( Y \) is complete, there exists a mapping \( A(x) : U \rightarrow V \) such that
\[
A(x) = \lim_{n \to \infty} \frac{g(2^n x)}{2^n} \forall x \in U.
\]
Letting \( n \to \infty \) in (31) and using (5), we see that (18) holds for all \( x \in U \). To show that \( A \) satisfies (3), replacing \( (x, y) \) by \( (2^n x, 2^n y) \) and dividing by \( 2^n \) in (17), we obtain
\[
\frac{1}{2^n} \mathcal{P}(2^n x, 2^n y) \leq \frac{1}{2^n} \alpha(2^n x, 2^n y)
\]
for all \( x, y \in X \). Letting \( n \to \infty \) in the above inequality and using the definition of \( A(x) \), we see that
\[
A(2x + y) - A(2x - y) = 4[A(x + y) - A(x - y)] - 6A(y).
\]
Hence \( A \) satisfies (3) for all \( x, y \in X \). To prove \( A \) is unique additive function satisfying (3), we let \( B(x) \) be another additive mapping satisfying (3) and (18), then
\[ \| A(x) - B(x) \| = \frac{1}{2^n} \| A(2^n x) - B(2^n x) \| \]
\[ \leq \frac{1}{2^n} \left\| A(2^n x) - f(2^n x) + 8 f(2^n x) \right\| + \| f(2^n x) - 8 f(2^n x) - B(2^n x) \| \]
\[ \leq \sum_{k=0}^{\infty} \frac{\beta(2^{k+n} x)}{2^{(k+n)}} \]
\[ \to 0 \text{ as } n \to \infty \]

for all \( x \in X \). Hence \( A \) is unique.

For \( j = -1 \), we can prove a similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [6], Ulam-Hyers-Rassias [14], Ulam-Gavruta-Rassias [11] and Ulam-JRassias [16] stabilities of (3).

**Corollary 3.2:** Let \( F : U \to V \) be a mapping and there exits real numbers \( \lambda \) and \( s \) such that

\[ \| F(x,y) \| \leq \begin{cases} 
\lambda x + \lambda y, & s < 1 \\
\lambda x^s + \lambda y^s, & s > 1 \\
\lambda x^s + \lambda y^s + \lambda x^s + \lambda y^s, & s > \frac{1}{2} \\
\lambda x^s + \lambda y^s + \lambda x^s + \lambda y^s, & s > \frac{1}{2} \\
\end{cases} \]

for all \( x, y \in U \), then there exists a unique additive function \( A : U \to V \) such that

\[ \| f(2x) - 8 f(x) - A(x) \| \leq \begin{cases} 
5 \lambda x, & 2 - 2^s \\
(18 + 2^{s+1}) \lambda x^s, & 2 - 2^s \\
(4 + 2^s) \lambda x^s, & 2 - 2^s \\
(22 + 2^s + 2 \cdot 2^s) \lambda x^s, & 2 - 2^s \\
\end{cases} \]

for all \( x \in U \).

**IV. Stability Results: Cubic Case**

In this section, we discussed the generalized Ulam-Hyers stability of the functional equation (3) for cubic case.

**Theorem 4.1:** Let \( j = \pm 1 \). Let \( F : U \to V \) be a mapping for which there exist a function \( \alpha : U \to [0, \infty) \) with the condition

\[ \lim_{n \to \infty} \frac{1}{8^n} \alpha(2^n x, 2^n y) = 0 \]

such that the functional inequality

\[ \| F(x,y) \| \leq \alpha(x,y) \]

for all \( x, y, z, w \in U \). Then there exists a unique cubic mapping \( C : U \to V \) satisfying the functional equation (3) and
\[ \|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{\beta_k} \beta(2^k x) \]  

(36)

for all \( x \in U \), where \( \beta(2^k x) \) and \( C(x) \) are defined by

\[ \beta(2^k x) = 4 \alpha(2^k x, 2^k x) + \alpha(2^k x, 2^{(k+1)} x) \]

(37)

and

\[ C(x) = \lim_{n \to \infty} \frac{1}{8^n} (f(2^{(n+1)} x) - 2f(2^n x)) \]

(38)

for all \( x \in U \), respectively.

**Proof.** It is easy to see from (24) that

\[ \|f(4x) - 2f(2x) - 8(f(2x) - 2f(x))\| \leq \beta(x) \]

(39)

for all \( x \in U \). Using (12) in (39), we obtain

\[ \|h(2x) - 8h(x)\| \leq \beta(x) \]

(40)

for all \( x \in U \). The rest proof is similar tracing to that of Theorem 3.1.

The following Corollary is an immediate consequence of Theorem 4.1 concerning the Ulam-Hyers [6], Ulam-Hyers-Rassias [14], Ulam-Gavruta-Rassias [11] and Ulam-J-Rassias [16] stabilities of (3).

**Corollary 4.2:** Let \( F : U \to V \) be a mapping and there exits real numbers \( \lambda \) and \( s \) such that

\[ \|F(x, y)\| \leq \begin{cases} 
\lambda, & s < \frac{3}{2} \quad \text{or} \quad s > \frac{3}{2}; \\
\lambda \|x\|^r + \|y\|^r, & s < 3 \quad \text{or} \quad s > 3; \\
\lambda \|x\|^r \|y\|^r, & s < \frac{3}{2} \quad \text{or} \quad s > \frac{3}{2}; \\
\lambda \|x\|^r \|y\|^r + \|x\|^{2s} + \|y\|^{2s} \|, & s < \frac{3}{2} \quad \text{or} \quad s > \frac{3}{2};
\end{cases} \]

(41)

for all \( x, y \in U \), then there exists a unique cubic function \( C : U \to V \) such that

\[ \|f(2x) - 2f(x) - C(x)\| \leq \begin{cases} 
\frac{5\lambda}{7}, & s \leq \frac{1}{2} \\
\frac{(18 + 2^{s+1})\lambda \|x\|^s}{|8 - 2^s|}, & s > \frac{1}{2} \quad \text{or} \quad s > \frac{3}{2} \\
\frac{(4 + 2^{2s})\lambda \|x\|^{2s}}{|8 - 2^{2s}|}, & s < \frac{3}{2} \quad \text{or} \quad s > \frac{3}{2} \\
\frac{(22 + 2^{2s} + 2 \cdot 2^{2s})\lambda \|x\|^{2s}}{|8 - 2^{2s}|}, & s < \frac{3}{2};
\end{cases} \]

(42)

for all \( x \in U \).

**V. STABILITY RESULTS: MIXED CASE**

In this section, we established the generalized Ulam-Hyers stability of the functional equation (3) for mixed case.

**Theorem 5.1:** Let \( j = \pm 1 \). Let \( F : U \to V \) be a mapping for which there exist a function \( \alpha : U^2 \to [0, \infty) \) with the condition given in (16) and (34) respectively, such that the functional inequality

\[ \|F(x, y)\| \leq \alpha(x, y) \]

(43)
for all \( x, y \in U \). Then there exists a unique additive mapping \( A : U \to V \) and a unique cubic mapping \( C : U \to V \) satisfying the functional equation (3) and

\[
\| f(x) - A(x) - C(x) \| \leq \frac{1}{6} \left\{ \frac{1}{2} \sum_{k=\frac{1}{2}}^{\infty} \beta(2^k x) + \frac{1}{8} \sum_{k=\frac{1}{2}}^{\infty} \beta(2^{2k} x) \right\}
\]

(44)

for all \( x \in U \). The mapping \( \beta(2^k x), A(x) \) and \( C(x) \) are respectively defined in (19), (20) and (38) for all \( x \in U \).

**Proof.** By Theorems 3.1 and 4.1, there exists a unique additive function \( A_i : U \to V \) and a unique cubic function \( C_i : U \to V \) such that

\[
\| f(2x) - 8f(x) - A_i(x) \| \leq \frac{1}{6} \sum_{k=\frac{1}{2}}^{\infty} \beta(2^{2k} x)
\]

(45)

and

\[
\| f(2x) - 2f(x) - C_i(x) \| \leq \frac{1}{8} \sum_{k=\frac{1}{2}}^{\infty} \beta(2^{2k} x)
\]

(46)

for all \( x \in U \). Now from (45) and (46), one can see that

\[
\| f(x) + \frac{1}{6} A_i(x) - \frac{1}{6} C_i(x) \| \leq \frac{1}{6} \left\{ \| f(2x) - 8f(x) - A_i(x) \| + \| f(2x) - 2f(x) - C_i(x) \| \right\}
\]

\[
\leq \frac{1}{6} \left\{ \frac{1}{2} \sum_{k=\frac{1}{2}}^{\infty} \beta(2^k x) + \frac{1}{8} \sum_{k=\frac{1}{2}}^{\infty} \beta(2^{2k} x) \right\}
\]

for all \( x \in U \). Thus we obtain (46) by defining \( A(x) = -\frac{1}{6} A_i(x) \) and \( C(x) = \frac{1}{6} C_i(x), \beta(2^{2k} x), A(x) \) and \( C(x) \) are respectively defined in (19), (20) and (38) for all \( x \in U \).

The following corollary is the immediate consequence of Theorem 5.1, using Corollaries 3.2 and 4.2 concerning the Ulam-Hyers [6], Ulam-Hyers-Rassias [14], Ulam-Gavruta-Rassias [11] and Ulam-J-Rassias [16] stabilities of (3).

**Corollary 5.2:** Let \( F : U \to V \) be a mapping and there exit real numbers \( \lambda \) and \( s \) such that

\[
\| F(x, y) \| \leq \begin{cases} 
\lambda, & s < 1 \quad \text{or} \quad s > 1; \\
\lambda \| x \| + \| y \|, & s < \frac{1}{2} \quad \text{or} \quad s > \frac{1}{2}; \\
\lambda \| x \|^{\| y \|} + \frac{1}{\| x \|^{2s} + \| y \|^{2s}}, & s < \frac{1}{2} \quad \text{or} \quad s > \frac{1}{2}; 
\end{cases}
\]

(47)

for all \( x, y \in U \), then there exists a unique additive mapping \( A : U \to V \) and a unique cubic mapping \( C : U \to V \) such that
\[ \| f(x) - A(x) - C(x) \| \leq \begin{cases} 
  \frac{5\lambda}{6} \left( 1 + \frac{1}{7} \right), \\
  \frac{(18+2^{s+1})}{6} \left( \frac{1}{|2-2^s|} + \frac{1}{|8-2^s|} \right) \lambda \| x \|^{s}, \\
  \frac{(4+2^{s})}{6} \left( \frac{1}{|2-2^{2s}|} + \frac{1}{|8-2^{2s}|} \right) \lambda \| x \|^{2s}, \\
  \frac{(22+2^{s}+2\cdot2^{s})}{6} \left( \frac{1}{|2-2^{2s}|} + \frac{1}{|8-2^{2s}|} \right) \lambda \| x \|^{2s} 
 \end{cases} \] (48)

for all \( x \in U \).

\section*{References}


