IMMOVABILITY OF CAUCHY ADDITIVE FUNCTIONAL EQUATION IN VARIOUS $\varphi-2-$ NORMED SPACES

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ABSTRACT. In this paper, the authors established the generalized Ulam - Hyers stability of the Cauchy Additive functional equation in $\varphi-2-$ normed space, Generalized $\varphi-2-$ normed space, Non - Archimedean Fuzzy $\varphi-2-$ normed spaces.

1. INTRODUCTION

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y).$$

(1.1)

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

During the last seven decades, the perturbation problems of several functional equations have been extensively investigated by number of authors [1, 2, 9, 11, 15, 16, 18, 19]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [4, 7, 12, 13, 14, 17].

The solution and stability of various additive functional equations were introduced and discussed in [3, 6, 5, 8] and references cited there in.

In this paper, the authors established the generalized Ulam - Hyers stability of the Cauchy additive functional equation

$$g(x+y) = g(x) + g(y)$$

(1.2)

in $\varphi-2-$ normed space, Generalized $\varphi-2-$ normed space, Non - Archimedean Fuzzy $\varphi-2-$ normed spaces.

Based on [10], some basic definitions and notations in $\varphi-2-$ normed spaces is provided.

Definition 1.1 A $t-$ norm $\triangledown$ is a two place function $\triangledown : [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non decreasing in each place and such that $a \triangledown 1 = a$, for all $a \in [0,1]$.

Definition 1.2 Let $\varphi$ be a function defined on the real field $\mathbb{R}$ into itself with the following properties :

$\cdot$ $\varphi(-t) = \varphi(t)$, for every $t \in \mathbb{R}$;
$\cdot$ $\varphi(1) = 1$;
$\cdot$ $\varphi$ is strict increasing and continuous on $(0, \infty)$;
$\cdot$ $\lim_{\alpha \to 0} \varphi(\alpha) = 0$ and $\lim_{\alpha \to \infty} \varphi(\alpha) = \infty$.

Example 1.3 The functions

$\cdot$ $\varphi(\alpha) = |\alpha|$ for every $\alpha \in \mathbb{R}$;
$\cdot$ $\varphi(\alpha^p) = |\alpha|^p$ for every $p \in \mathbb{R}_+$. 

In Section 2, 3 and 4 the generalized Ulam - Hyers stability of the Cauchy additive functional equation (1.2) in $\varphi-2-$ normed space, Generalized $\varphi-2-$ normed space and Non - Archimedean Fuzzy $\varphi-2-$ normed space are respectively discussed.
2. STABILITY RESULT: $\varphi - 2$– NORMED SPACE

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.2) in $\varphi - 2$– normed space.

Now, some basic definitions related to $\varphi - 2$– normed spaces is given.

**Definition 2.1** Let $X$ be linear space. A function $\|.,.\| : X \times X \rightarrow [0, \infty)$ is called a $\varphi - 2$– normed space if it satisfies to following

- (2N1) $\| x, y \| = 0$ if and only if $x$ and $y$ are linearly independent vectors.
- (2N2) $\| x, y \| = \| y, x \|$ for all $x, y \in X$.
- (2N3) $\| \lambda x, y \| = \| \varphi(\lambda) \| \| x, y \|$ for all $x, y \in X$.
- (2N4) $\| x + y, z \| \leq \| x, z \| + \| y, z \|$ for all $x, y, z \in X$.

The $\varphi - 2$– normed space is denoted by $(X, \|.,.\|)$.

**Definition 2.2** A sequence $\{x_n\}$ in a $\varphi - 2$– normed space $(X, \|.,.\|)$ if called convergent if there exist $x \in X$ such that $\lim_{n \to \infty} \| x_n - x \| = 0$ then $\lim_{n \to \infty} \| x_n, y \| = \| x, y \|$ for all $y \in X$.

**Definition 2.3** A sequence $\{x_n\}$ in a $\varphi - 2$– normed space $(X, \|.,.\|)$ is called Cauchy sequence if $\| x_n, x \|$ is convergent.

**Definition 2.4** A $\varphi - 2$– normed space $(X, \|.,.\|)$ is called $\varphi - 2$– Banach space if every Cauchy sequence is convergent.

Throughout this section let $(X, \|.,.\|)$ be a $\varphi - 2$– normed space and $(Y, \|.,.\|)$ be a $\varphi - 2$– Banach space.

**Theorem 2.5** Let $j = \pm 1$. Let $g : X \rightarrow Y$ be a mapping for which there exist a function $\alpha : X^2 \rightarrow [0, \infty)$ with the condition

$$\lim_{n \to \infty} \frac{1}{2^n} \alpha\left(2^n x, s\right)(2^n y, s) = 0$$

such that the functional inequality

$$\| g(x + y) - g(x) - g(y) \| \leq \alpha((x, s), (y, s))$$

for all $x, y \in X$ and all $s \in X$. Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying the functional equation (1.2) and

$$\| g(x) - A(x), s \| \leq \frac{1}{\varphi(2)} \sum_{k=-N}^{\infty} \delta(2^k x, s) \frac{2^k}{2^j}$$

where

$$\delta(2^k x, s) = \alpha((2^k x, s), (2^k x, s))$$

for all $x \in X$ and all $s \in X$. The mapping $A(x)$ is defined by

$$\lim_{n \to \infty} \| g(2^n x), s \| \frac{2^n}{2^j} = \| A(x), s \|$$

for all $x \in X$ and all $s \in X$.

Proof. Assume $j = 1$. Replacing $(x, y)$ by $(x, x)$ in (2.2) and using (2N3), we get

$$\| g(x) - \frac{g(2x)}{2}, s \| \leq \frac{1}{\varphi(2)} \alpha((x, s), (x, s))$$
for all \( x \in X \) and all \( s \in X \). It follows from (2.6) that
\[
\left\| g(x) - \frac{g(2x)}{2}, s \right\| \leq \delta((x,s)) / \phi(2) \tag{2.7}
\]
where
\[
\delta((x,s)) = \alpha((x,s),(x,s))
\]
for all \( x \in X \) and all \( s \in X \). Now replacing \( x \) by \( 2x \) and dividing by \( 2 \) in (2.7), we obtain
\[
\left\| g(2x) - g(2^2x), s \right\| \leq \delta(2x,s) / \phi(2) \cdot 2 \tag{2.8}
\]
for all \( x \in X \) and all \( s \in X \). It follows from (2.7), (2.8) and (2N.4), we have
\[
\left\| g(x) - \frac{g(2^2x)}{2^2}, s \right\| \leq \left\| g(x) - \frac{g(2x)}{2}, s \right\| + \left\| g(2x) - \frac{g(2^2x)}{2^2}, s \right\| \leq \frac{1}{\phi(2)} \left( \delta(x,s) + \delta(2x,s) / 2 \right) \tag{2.9}
\]
for all \( x \in X \) and all \( s \in X \). Proceeding further and using induction on a positive integer \( n \), we get
\[
\left\| g(x) - \frac{g(2^n x)}{2^n}, s \right\| \leq \frac{1}{\phi(2)} \sum_{k=0}^{n-1} \delta(2^k x,s) / 2^k \leq \frac{1}{\phi(2)} \sum_{k=0}^{\infty} \delta(2^k x,s) / 2^k \tag{2.10}
\]
for all \( x \in X \) and all \( s \in X \). In order to prove the convergence of the sequence \( \left\{ \frac{g(2^n x)}{2^n} \right\} \), replace \( x \) by \( 2^n x \) and dividing by \( 2^n \) in (2.10), for any \( m,n > 0 \), we deduce
\[
\left\| g(2^m x) - g(2^{n+m} x), s \right\| = \frac{1}{2^m} \left\| g(2^m x) - g(2^n \cdot 2^n x), s \right\| \leq \frac{1}{\phi(2)} \sum_{k=0}^{\infty} \delta(2^k x,s) / 2^k \to 0 \text{ as } m \to \infty
\]
for all \( x \in X \) and all \( s \in X \). Hence the sequence \( \left\{ \frac{g(2^n x)}{2^n} \right\} \) is a Cauchy sequence. Since \( Y \) is complete, there exists a mapping \( A : X \to Y \) such that
\[
\lim_{n \to \infty} \left\| \frac{g(2^n x)}{2^n}, s \right\| = \left\| A(x), s \right\| \quad \forall \ x \in X, \ s \in X
\]
Letting \( n \to \infty \) in (2.10), we see that (2.3) holds for all \( x \in X \). To prove that \( A \) satisfies (1.2), replacing \( (x,y) \) by \( (2^n x, 2^n y) \) and dividing by \( 2^n \) in (2.2), we obtain
\[
\frac{1}{2^n} \left\| g(2^n (x+y)) - g(2^n x) - g(2^n y), s \right\| \leq \frac{1}{2^n} \alpha((2^n x, s), (2^n y, s))
\]
for all \( x, y \in X \) and all \( s \in X \). Letting \( n \to \infty \) in the above inequality, using (2.1) and the definition of \( A(x) \) also (2N.1), we see that \( A \) satisfies (1.2) for all \( x, y \in X \). To prove that \( A(x) \) is unique, let \( B(x) \) be another additive mapping satisfying (1.2) and (2.3), then
\[
\left\| A(x) - B(x), s \right\| \leq \frac{1}{2^n} \left\| A(2^n x) - g(2^n x), s \right\| + \left\| g(2^n x) - B(2^n x), s \right\| \leq \frac{2}{\phi(2)} \sum_{k=0}^{\infty} \delta(2^{k+n} x,s) / 2^{(k+n)} \to 0 \text{ as } n \to \infty
\]
for all \( x \in X \) and all \( s \in X \). Hence \( A \) is unique.

For \( j = -1 \), we can prove a similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 2.5 concerning the Ulam-Hyers [11], Ulam-TRassias [16], Ulam-GRassias [15] and Ulam-JRassias [18] stabilities of (1.2).
Corollary 2.6 Let \( g : X \to Y \) be a function and there exists real numbers \( \lambda \) and \( t \) such that

\[
\|g(x + y) - g(x) - g(y),s\| \leq \begin{cases} 
\lambda, & t < 1 \quad \text{or} \quad t > 1; \\
\lambda \left\{ \|x,s\| + \|y,s\| \right\}, & 2t < 1 \quad \text{or} \quad 2t > 1; \\
\lambda \left\{ \|x,s\| \|y,s\| \right\}, & 2t < 1 \quad \text{or} \quad 2t > 1; \\
\lambda \left\{ \|x,s\| \|y,s\| + \left\{ \|x,s\|^{2t} + \|y,s\|^{2t} \right\} \right\}, & 2t < 1 \quad \text{or} \quad 2t > 1;
\end{cases}
\]

for all \( x,y \in X \) and all \( s \in X \). Then there exists a unique additive function \( A : X \to Y \) such that

\[
\|g(x) - A(x),s\| \leq \begin{cases} 
\frac{2\lambda}{\varphi(2)}, & \varphi(2) > 2t; \\
\frac{4\lambda \|x,s\|}{\varphi(2) - 2t}, & \varphi(2) > 2t; \\
\frac{\lambda \|x,s\|^{2t}}{\varphi(2) - 2t}, & \varphi(2) > 2t; \\
\frac{5\lambda \|x,s\|^{2t}}{\varphi(2) - 2t}, & \varphi(2) > 2t;
\end{cases}
\]

for all \( x \in X \) and all \( s \in X \).

3. STABILITY RESULT: GENERALIZED \( \varphi - 2 \) – NORMED SPACE

In this section, the generalized Ulam - Hyers stability of the functional equation (1.2) in Generalized \( \varphi - 2 \) – normed space is given.

Now, some basic definitions related to Generalized \( \varphi - 2 \) – normed spaces is provided.

Definition 3.1 Let \( X \) be linear space. A function \( N(\cdot,\cdot) : X \times X \to \{0,\infty\} \) is called a generalized \( \varphi - 2 \) – normed space if it satisfies the following

(2G1) \( N(x,y) = 0 \) if and only if \( x \) and \( y \) are linearly independent vectors.

(2G2) \( N(x,y) = N(y,x) \) for all \( x,y \in X \),

(2G3) \( N(\lambda x,y) = |\varphi(\lambda)| N(x,y) \) for all \( x,y \in X \),

(2G4) \( N(x+y,z) \leq N(x,z) + N(y,z) \) for all \( x,y,z \in X \).

The generalized \( \varphi - 2 \) – normed space is denoted by \( (X,N(\cdot,\cdot)) \).

Definition 3.2 A sequence \( \{x_n\} \) in a generalized \( \varphi - 2 \) – normed space \( (X,N(\cdot,\cdot)) \) is called convergent if there exist \( x \in X \) such that \( \lim_{n \to \infty} N(x_n,x,y) = 0 \) then \( \lim_{n \to \infty} N(x_n,y) = N(x,y) \) for all \( y \in X \).

Definition 3.3 A sequence \( \{x_n\} \) in a generalized \( \varphi - 2 \) – normed space \( (X,N(\cdot,\cdot)) \) is called Cauchy sequence if there exist two linearly independent elements \( y \) and \( z \) in \( X \) such that \( \{N(x_n,y)\} \) and \( \{N(x_n,z)\} \) are real Cauchy sequences.

Definition 3.4 A generalized \( \varphi - 2 \) – normed space \( (X,N(\cdot,\cdot)) \) is called generalized \( \varphi - 2 \) – Banach space if every Cauchy sequence is convergent.

Now let us consider \( X \) be a generalized \( \varphi - 2 \) – normed space and \( Y \) be generalized \( \varphi - 2 \) – Banach space, respectively.
Theorem 3.5 Let \( j = \pm 1 \). Let \( g : X \to Y \) be a mapping for which there exist a function \( \alpha, \delta : X^2 \to [0, \infty) \) with the condition
\[
\lim_{n \to \infty} \frac{1}{2^n} \alpha((2^n x, s), (2^n y, s)) = 0
\]
and the functional inequality
\[
N(g(x + y) - g(x) - g(y), s) \leq \alpha((x, s), (y, s))
\]
for all \( x, y \in X \) and all \( s \in X \). Then there exists a unique additive mapping \( A : X \to Y \) satisfying the functional equation (1.2) and
\[
N(g(x) - A(x), s) \leq \frac{1}{\varphi(2)} \sum_{k=\lfloor \log_2 s \rfloor}^{\infty} \delta(2^k x, s)
\]
where
\[
\delta(2^k x, s) = \alpha((2^k x, s), (2^k x, s))
\]
for all \( x \in X \) and all \( s \in X \). The mapping \( A(x) \) is defined by
\[
\lim_{n \to \infty} N\left(\frac{g(2^n x)}{2^n}, s\right) = N(A(x), s)
\]
for all \( x \in X \) and all \( s \in X \).

Proof. Assume \( j = 1 \). Replacing \( (x, y) \) by \( (x, x) \) in (3.2) and using (2G3), we get
\[
N\left(\frac{g(2^n x)}{2^n}, s\right) \leq \delta((x, s))
\]
where \( \delta((x, s)) = \alpha((x, s), (x, s)) \) for all \( x \in X \) and all \( s \in X \). The rest of the proof is similar to that of Theorem 2.5.

The following Corollary is an immediate consequence of Theorem 3.5 concerning the Ulam-Hyers [11], Ulam-Rassias [16], Ulam-GRassias [15] and Ulam-JRassias [18] stabilities of (1.2).

Corollary 3.6 Let \( g : X \to Y \) be a function and there exits real numbers \( \lambda \) and \( \varphi \) such that
\[
N(g(x + y) - g(x) - g(y), s) \leq \begin{cases}
\lambda, & t < 1 \ 	ext{or} \ t > 1; \\
\lambda \|x, s\| + \|y, s\|, & 2t < 1 \ 	ext{or} \ 2t > 1; \\
\lambda \|x, s\| + \|y, s\| + \|x, s\|^{2t} + \|y, s\|^{2t}, & 2t < 1 \ 	ext{or} \ 2t > 1;
\end{cases}
\]
for all \( x, y \in X \) and all \( s \in X \). Then there exists a unique additive function \( A : X \to Y \) such that
\[
N(g(x) - A(x), s) \leq \begin{cases}
\frac{2\lambda}{\varphi(2)}, & \\
\frac{4\lambda \|x, s\|^t}{\varphi(2) - 2^t}, & \\
\frac{\lambda \|x, s\|^{2t}}{\varphi(2) - 2^{2t}}, & \\
\frac{5\lambda \|x, s\|^{2t}}{\varphi(2) - 2^{2t}}, &
\end{cases}
\]
for all \( x \in X \) and all \( s \in X \).
4. STABILITY RESULT: NON-ARCHIMEDEAN FUZZY $\varphi - 2 -$ NORMED SPACE

In this section, the generalized Ulam - Hyers stability of the additive functional equation (1.2) non-Archimedean fuzzy $\varphi - 2 -$ normed space is provided.

Now, some definitions and notations in non-Archimedean fuzzy $\varphi - 2 -$ normed space is present.

**Definition 4.1** [10] Let $L$ be a linear space over the field $\mathbb{R}$ of a dimension greater than one and let $N$ be a mapping defined on $L \times L \times [0, \infty)$ with values into $[0,1]$ satisfying the following conditions: for all $x,y,z \in L$ and $s,t \in [0, \infty)$

(2NAF1) $N(x,y,0) = 0$ ;
(2NAF2) $N(x,y,t) = 1$, for all $t > 0$ if and only if $x,y$ are linear dependent;
(2NAF3) $N(x,y,t) = N(y,x,t)$ for all $x,y \in L$, and $t > 0$ ;
(2NAF4) $N(x + y, z, \max(t,s)) \geq \min(N(x,z,t) \circ N(y,z,s))$;
(2NAF5) $N(x,y,\cdot) : [0, \infty) \to [0,1]$ is left continuous.

(2NAF6) $N(\alpha x,y,t) = N\left(x,y,\frac{t}{\varphi(\alpha)}\right), \alpha \in \mathbb{R}$.

The triple $(L,N,\circ)$ will be called a non-Archimedean fuzzy $\varphi - 2 -$ normed space.

**Example 4.2** Let $(L,\| \cdot \|, \cdot)$ be a non-Archimedean fuzzy $\varphi - 2 -$ normed space. Then

$$N(x,y,t) = \begin{cases} \frac{t}{t + \|x,y\|}, & t > 0, x,y \in X, \\ 0, & t \leq 0, x,y \in X. \end{cases}$$

Then $(L,N,\circ)$ is a non-Archimedean fuzzy $\varphi - 2 -$ normed space.

**Definition 4.3** Let $(L,N,\circ)$ be a non-Archimedean fuzzy $\varphi - 2 -$ normed space. Let $x_n$ be a sequence in $L$. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n \to \infty} N(x_n - x,a,t) = 1$$

for all $a \in L$ and $t > 0$. In that case, $x$ is called the limit of the sequence $x_n$ and we denote it by

$$N - \lim_{n \to \infty} x_n = x.$$

**Definition 4.4** A sequence $x_n$ in $L$ is called Cauchy if

$$N(x_{n+p} - x_n,a,t) = 1$$

for all $a \in L$, $p > 0$ and $t > 0$.

**Definition 4.5** Every convergent sequence in a non-Archimedean fuzzy $\varphi - 2 -$ normed space is a Cauchy sequence. If every Cauchy sequence is convergent, then the non-Archimedean fuzzy $\varphi - 2 -$ normed space is called a non-Archimedean fuzzy $\varphi - 2 -$ Banach space.

Hereafter, throughout this section, assume that $\mathbb{R}$ be a non-Archimedean field, $X$ be a vector space over $\mathbb{R}$, $(Y, N', \circ)$ be a non-Archimedean fuzzy $\varphi - 2 -$ Banach space over $\mathbb{R}$ and $(Z, N', \circ)$ be an non-Archimedean fuzzy $\varphi - 2 -$ normed space.

**Theorem 4.6** Let $\gamma \in \{-1,1\}$ be fixed and let $\alpha : X^2 \to Z$ be a mapping such that for some $\kappa$ with

$$0 < \left(\frac{\varphi(\kappa)}{\varphi(2)}\right)^\gamma < 1$$
\[ N'(\alpha \left( 2^r x, 2^r x \right), a, r) \geq N' \left( \kappa^\gamma \alpha \left( x, x \right), a, r \right) \]  \hspace{1cm} (4.1)

for all \( x, a \in X \) and all \( r > 0 \), and

\[ \lim_{n \to \infty} N' \left( \alpha \left( 2^n x, 2^n y, a, [\varphi(2^n)]^r \right) \right) = 1 \]

\hspace{1cm} (4.2)

for all \( x, y, a \in X \) and all \( r > 0 \). Suppose that a function \( g : X \to Y \) satisfies the inequality

\[ N(g(x + y) - g(x) - g(y), a, r) \geq N'(\alpha(x, y), a, r) \]

\hspace{1cm} (4.3)

for all \( x, y, a \in X \) and all \( r > 0 \). Then the limit

\[ A(x) = N - \lim_{n \to \infty} \frac{g(2^n x)}{2^n} \]

\hspace{1cm} (4.4)

exists for all \( x \in X \) and the mapping \( A : X \to Y \) is a unique additive mapping satisfying (1.2) and

\[ N(g(x) - A(x), a, r) \geq N' \left( \alpha(x, x), a, r \mid \varphi(2) - \varphi(\kappa) \right) \]

\hspace{1cm} (4.5)

for all \( x, a \in X \) and all \( r > 0 \).

Proof. First assume \( \gamma = 1 \). Replacing \( (x, y) \) by \( (x, x) \) in (4.3), we get

\[ N(g(2x) - 2g(x), a, r) \geq N'(\alpha(x, x), a, r) \]

\hspace{1cm} (4.6)

for all \( x, a \in X \) and all \( r > 0 \). Replacing \( x \) by \( 2^n x \) in (4.6) and using \((2NAF6)\), we obtain

\[ N \left( \frac{g(2^{n+1} x)}{2} - g(2^n x), a, \frac{r}{\varphi(2)} \right) \geq N'(\alpha(2^n x, 2^n x), a, r) \]

\hspace{1cm} (4.7)

for all \( x, a \in X \) and all \( r > 0 \). Using (4.1), \((2NAF6)\) in (4.7), we arrive

\[ N \left( \frac{g(2^{n+1} x)}{2} - g(2^n x), a, \frac{r}{\varphi(2) \cdot \varphi(2^n)} \right) \geq N' \left( \alpha(x, x), a, \frac{r}{\varphi(\kappa^n)} \right) \]

\hspace{1cm} (4.8)

for all \( x, a \in X \) and all \( r > 0 \). It follows from (4.8), that

\[ N \left( \frac{g(2^{n+1} x)}{2} - g(2^n x), a, \frac{\varphi(\kappa^n) r}{\varphi(2) \cdot \varphi(2^n)} \right) \geq N'(\alpha(x, x), a, r) \]

\hspace{1cm} (4.9)

holds for all \( x, a \in X \) and all \( r > 0 \). Replacing \( r \) by \( \varphi(\kappa^n) r \) in (4.9), we get

\[ N \left( \frac{g(2^{n+1} x)}{2} - g(2^n x), a, \frac{\varphi(\kappa^n) r}{\varphi(2) \cdot \varphi(2^n)} \right) \geq N'(\alpha(x, x), a, r) \]

\hspace{1cm} (4.10)

for all \( x, a \in X \) and all \( r > 0 \). It is easy to verify that

\[ \frac{g(2^n x)}{2^n} - g(x) = \sum_{i=0}^{n-1} \frac{g(2^{i+1} x)}{2^{i+1}} - \frac{g(2^i x)}{2^i} \]

\hspace{1cm} (4.11)

for all \( x \in X \). From equations (4.10) and (4.11), we have

\[ N \left( \frac{g(2^n x)}{2^n} - g(x), a, \sum_{i=0}^{n-1} \frac{\varphi(\kappa) i}{\varphi(2)} \right) \geq \min_{i=0}^{n-1} \left\{ N \left( \frac{g(2^{i+1} x)}{2^{i+1}} - g(2^i x), a, \frac{\varphi(\kappa) i}{\varphi(2)} \right) \right\} \]

\[ \geq \min_{i=0}^{n-1} \left\{ N'(\alpha(x, x), a, r) \right\} \geq N'(\alpha(x, x), a, r) \]

\hspace{1cm} (4.12)

for all \( x, a \in X \) and all \( r > 0 \). Replacing \( x \) by \( 2^m x \) in (4.12) and using (4.1), \((2NAF6)\), we obtain

\[ N \left( \frac{g(2^{n+m} x)}{2^{n+m}} - g(2^m x), a, \sum_{i=0}^{n-1} \frac{\varphi(\kappa) i}{\varphi(2)^{i+1}} \cdot \frac{r}{\varphi(2^m)} \right) \geq N'(\alpha(x, x), a, \frac{r}{\varphi(\kappa)^m}) \]

\hspace{1cm} (4.13)
for all \( x, a \in X \) and all \( r > 0 \) and all \( m, n \geq 0 \). Replacing \( r \) by \( \varphi(\kappa^m)r \) in (4.13), we get

\[
N\left( \frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^nx)}{2^m}, a, \sum_{i=m}^{m+n-1} \frac{[\varphi(\kappa)]^m}{[\varphi(2)]^{i+1}} \right) \geq N'\left( \alpha(x,x), a, r \right)
\]  

(4.14)

for all \( x, a \in X \) and all \( r > 0 \) and all \( m, n \geq 0 \). It follows from (4.14) that

\[
N\left( \frac{g(2^{n+m}x)}{2^{n+m}} - \frac{g(2^nx)}{2^m}, a, r \right) \geq N'\left( \alpha(x,x), a, r, \sum_{i=m}^{m+n-1} \frac{[\varphi(\kappa)]^m}{[\varphi(2)]^{i+1}} \right)
\]  

(4.15)

for all \( x, a \in X \) and all \( r > 0 \) and all \( m, n \geq 0 \). Since \( \varphi(2) < \infty \) and \( \varphi(2) < \infty \), using (4.15) implies that \( \left\{ \frac{g(2^nx)}{2^n} \right\} \) is a Cauchy sequence in \((Y,N')\). Since \((Y,N')\) is a non-Archimedean fuzzy Banach space, this sequence converges to some point \( A(x) \in Y \) such that

\[
A(x) = \lim_{n \to \infty} \frac{g(2^nx)}{2^n}
\]

for all \( x \in X \). Letting \( m = 0 \) in (4.15), we get

\[
N\left( \frac{g(2^nx)}{2^n} - g(x), a, r \right) \geq N'\left( \alpha(x,x), a, r, \sum_{i=0}^{m-1} \frac{[\varphi(\kappa)]^m}{[\varphi(2)]^{i+1}} \right)
\]  

(4.16)

for all \( x, a \in X \) and all \( r > 0 \). Letting \( n \to \infty \) in (4.16) and using (2NAF5), we arrive

\[
N\left( g(x) - A(x), a, r \right) \geq N'\left( \alpha(x,x), a, r(\varphi(2) - \varphi(\kappa)) \right)
\]

for all \( x, a \in X \) and all \( r > 0 \). To prove \( A \) satisfies (4.2), replacing \((x,y)\) by \((2^nx,2^ny)\) in (4.3), respectively, we obtain

\[
N\left( \frac{1}{2^n}[g(2^n(x+y) - g(2^nx) - g(2^ny)], a, r \right) \geq N'\left( \alpha(2^nx,2^ny), a, \varphi(2^n)r \right)
\]  

(4.17)

for all \( x, y \in X \) and all \( r > 0 \). Now,

\[
N\left( A(x+y) - A(x) - A(y), a, r \right)
\]

\[
\geq \min \left\{ N\left( A(x+y) - \frac{1}{2^n} g(2^n(x+y)], a, r \right), N\left( -A(x) + \frac{1}{2^n} g(2^nx), a, r \right), \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
\[ N(A(x) - A'(x), a, r) \geq \min \left\{ N\left( \frac{A(2^n x) - g(2^n x)}{2^n}, a, \frac{r}{2} \right), N\left( \frac{g(2^n x) - A'(2^n x)}{2^n}, a, \frac{r}{2} \right) \right\} \]

\[ \geq N'\left( \alpha(x, x), a, \frac{r \varphi(2^n)(\varphi(2) - \varphi(\kappa))}{2 \varphi(\kappa^n)} \right) \]

for all \( x, a \in X \) and all \( r > 0 \). Since \( \lim_{n \to \infty} \frac{r \varphi(2^n)(\varphi(2) - \varphi(\kappa))}{2 \varphi(\kappa^n)} = \infty \), we obtain

\[ \lim_{n \to \infty} N'\left( \alpha(x, x), a, \frac{r \varphi(2^n)(\varphi(2) - \varphi(\kappa))}{2 \varphi(\kappa^n)} \right) = 1. \]

Thus

\[ N(A(x) - A'(x), a, r) = 1 \]

for all \( x, a \in X \) and all \( r > 0 \), hence \( A(x) = A'(x) \). Therefore \( A(x) \) is unique.

For \( \gamma = -1 \), we can prove the result by a similar method. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 4.6 concerning the Ulam-Hyers [11], Ulam-TRassias [16], Ulam-GRassias [15] and Ulam-JRassias [18] stabilities of (4.2).

**Corollary 4.7** Suppose that a function \( g : X \to Y \) satisfies the inequality

\[ N(g(x + y) - g(x) - g(y), a, r) \geq \left\{ \begin{array}{ll} N'(\varepsilon, a, r), & \varepsilon = 0; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s < 1 \text{ or } d > 1; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s > 1 \text{ or } d < 1; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s > \frac{1}{2} \text{ or } d > \frac{1}{2}; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s < \frac{1}{2} \text{ or } d < \frac{1}{2}; \end{array} \right. \]  

(4.20)

for all \( x, y, a \in X \) and all \( r > 0 \), where \( \varepsilon, s, d \) are constants with \( \varepsilon > 0 \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[ N(g(x) - A(x), r) \geq \left\{ \begin{array}{ll} \varepsilon, & \varepsilon = 0; \\ \varepsilon \|x\|^s, & s < 1 \text{ or } s > 1; \\ \varepsilon \|x\|^s, & s > \frac{1}{2} \text{ or } s < \frac{1}{2}; \end{array} \right. \]  

(4.21)

for all \( x, a \in X \) and all \( r > 0 \).

Proof. If we define

\[ \alpha(x, y) = \left\{ \begin{array}{ll} N'(\varepsilon, a, r), & \varepsilon = 0; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s < 1 \text{ or } d > 1; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s > 1 \text{ or } d < 1; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s > \frac{1}{2} \text{ or } d > \frac{1}{2}; \\ N'(\varepsilon, \|x\|^s, \|y\|^d, a, r), & s < \frac{1}{2} \text{ or } d < \frac{1}{2}; \end{array} \right. \]

then the corollary is followed from Theorem 4.6, if

\[ d = \left\{ \begin{array}{ll} \varphi(0), & \varphi = 0; \\ \varphi(2^n), & n = 0 \text{ or } n > 0; \\ \varphi(2^n), & n > 0; \end{array} \right. \]
REFERENCES