Total Vertex Irregularity Strength of Circular Ladder and Windmill Graphs

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Abstract

Let $G(V,E)$ be a simple graph. For a total labeling $\partial : V \cup E \rightarrow \{1,2,3,...,k\}$ the weight of a vertex x is defined as $wt(x) = \partial(x) + \sum_{xy \in E} \partial(xy).$ ∂ is called a vertex irregular total *k*-labeling if for every pair of distinct vertices *x* and *y*, $wt(x) \neq wt(y).$. The minimum *k* for which the graph *G* has a vertex irregular total *k*-labeling is called the *total vertex irregularity strength* of *G* and is denoted by *tvs*(*G*). In this paper we determine the total vertex irregularity strength of circular ladder and windmill graphs.

Keywords: graph labeling; total vertex irregularity strength; weight of a vertex; circular ladder; windmill graph.

1. Introduction

Graph labelings were first introduced in the mid sixties. It concerns the assignment of values, usually represented by integers, to the edges and /or vertices of a graph. The intended purpose is to meet certain conditions within the graph. Many of the graph labeling methods were motivated by applications to technology and sports tournament scheduling. The existence of graphs for which a specified set of integer values is assigned to its nodes or edges or both according to some given criteria has been investigated in recent years. Such graphs are called *labeled graphs*. Labeled graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. They are also of interest in their own right due to their abstract mathematical properties arising from various structural considerations of the underlying graphs.

The qualitative labeling of graphs has inspired research in diverse fields of human enquiry such as conflict resolution in social psychology, electrical circuit theory and energy crisis. Quantitative labeling of graphs has led to quite intricate fields of applications such as coding theory problems including the design of good radar location codes, missile guidance codes and convolution codes with optimal auto correlation properties. The theoretical applications of labeled graphs are numerous, not only within the theory of graphs but also in other areas of mathematics such as combinatorial number theory, linear algebra and group theory.

2. Total Vertex Irregularity Strength of a Graph

A graph labeling is an assignment of labels, represented by integers, to the vertices, edges or both of a graph. Formally, given a graph *G*, a vertex labeling is a function mapping vertices of *G* to a set of integers. A graph with such a function defined is called a *vertex-labeled graph*. Likewise, an edge labeling is a function mapping edges of *G* to a set of labels. In this case, *G* is called an *edge-labeled graph*.

Motivated by the notion of the irregularity strength of a graph introduced by Chartrand et al. [8] and various kinds of other total labelings, Baca et al. [6] introduced the total vertex irregularity strength of a graph as follows: Let $G(V,E)$ be a simple graph. For a total labeling $\partial : V \cup E \rightarrow \{1,2,3,...,k\}$ the weight of a vertex *x* is defined as $wt(x) = \partial(x) + \sum_{xy \in E} \partial(xy).$ ∂ is called a vertex irregular total *k*-labeling if for every pair of distinct vertices *x* and *y*,

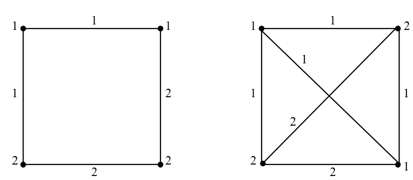
$wt(x) \neq wt(y)$. . The minimum k for which the graph G a vertex irregular total k-labeling is called the *total vertex irregularity strength of G* and is denoted by *tvs*(G).

For the survey on total vertex irregularity strength we refer to [10]. Baca et. al. [6] proved that $tvs(C_n) = \left\lceil \frac{n+2}{3} \right\rceil, n \geq 2; tvs(K_n) = 2; tvs(K_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil; tvs(C_n \times P_2) = \left\lceil \frac{2n+3}{4} \right\rceil$. If T is a tree with m pendent vertices and no of degree 2, they proved that $\left\lceil \frac{t+1}{2} \right\rceil \leq tvs(T) \leq m.$ They also proved that if G is a (p, q) graph with minimum degree δ and maximum degree Δ then $\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq tvs(T) \leq p + \Delta - 2\delta + 1.$

Ahmad A., Baskoro E. T. and Imran M. [3] determined the lower bound of total vertex irregularity strength of any graph and conjectured that the lower bound is tight. Ahmad, Ahtsham, H.B. Imran and A.Q.Gaig [1] determined the total vertex irregularity strength for five families of cubic plane graphs. Ahmad and Baca [4] proved that $tvs(J_{n,2}) = \left\lceil \frac{n+1}{2} \right\rceil, n > 4$ and conjectured that for $n \geq 3$ and $m \geq 3, tvs(J_{n,m}) = \max\left\{\left\lceil \frac{n(m-1)+2}{3} \right\rceil, \left\lceil \frac{nm+2}{4} \right\rceil\right\}$. They also proved that for the circulant graph $C_n(1,2), n \geq 5, tvs(C_n(1,2)) = \left\lceil \frac{n+4}{5} \right\rceil$. They also conjectured that $C_n(a_1, a_2, a_3, ..., a_r)$ with degree at least 5 and $n \geq 5, 1 \leq a_i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $tvs(C_n(a_1, a_2, a_3, ..., a_r)) = \left\lceil \frac{n+r}{1+r} \right\rceil$. In [3] Ahmad shows that the total vertex irregularity strength of the antiprism graph $A_n (n \geq 3)$ is $\left\lceil \frac{2n+4}{5} \right\rceil$. Furthermore, Ahmad et. al. [2, 4] found the total vertex irregularity strength for Jahangir graphs, circulant graphs, convex polytope and wheel related graphs. Ancholar, Kalkowski and Przybylo [5] proved that for every graph with $\delta(G) \geq 0, tvs(G) \leq \left\lceil \frac{3p}{\delta} \right\rceil + 1.$

Illustration

tvs(G_1) = 2 and *tvs*(G_2) = 2. See Figure 1 (a) &(b).



igure 1 (a) *tvs*(G_1) = 2; (b) *tvs*(G_2) = 2

The following is the key result used for proving the first result in this paper.

Theorem 2.1 [10]: Let G be an r-regular graph on n vertices. Then $\left\lceil \frac{n+r}{r+1} \right\rceil \leq tvs(G).$

3. Circular Ladder Graphs

In this section we determine the total vertex irregularity strength of circular ladder *CL*(*n*).

Definition 3.1 [9]: A circular ladder *CL*(n) is the union of an outer cycle $\Gamma_0 : u_1u_2u_3...u_nu_1$ and an inner cycle $\Gamma_1 : v_1v_2v_3...v_nv_1$ with additional edges $(u_i, v_i), i = 1, 2, 3, ..., n$ called spokes.

Theorem 3.1: $tvs(CL(n)) = \left\lceil \frac{2n+3}{4} \right\rceil.$

Proof : Let $CL(n)$ be the circular ladder with $V(CL(n)) = \{u_i, v_i, 1 \le i \le n\}$ taken in the anticlockwise order and $E(CL(n)) = \{e_i, 1 \le i \le n\} \cup \{g_i, 1 \le i \le n\} \cup \{h_i, 1 \le i \le n\}$ where $e_i = (u_i, u_{i+1}), 1 \le i \le n-1,$ $e_n = (u_n, u_1),$ $g_i = (v_i, v_{i+1}), 1 \le i \le n-1, g_n = (v_n, v_1)$ and $h_i = (u_i, v_i), 1 \le i \le n.$ Let k denote the number of spokes. Then clearly $k = \frac{n}{2}.$ The lower bound follows from Theorem 2.1. To show that $\left\lceil \frac{2n+3}{4} \right\rceil$ is an upper bound for $tvs(CL(n))$ we describe a total $\left\lceil \frac{2n+3}{4} \right\rceil$ labeling for $CL(n)$.For $n \ge 10$ we construct the function φ as follows:

Case 1: $k \equiv 0 (\bmod 4)$

$$\varphi(u_i) = \begin{cases} i & for \quad 1 \le i \le \frac{n}{2}, \\ n+1-i & for \quad \frac{n}{2}+1 \le i \le n, \end{cases} \quad \varphi(v_i) = \begin{cases} i & for \quad 1 \le i \le \frac{n}{2}+1, \\ n+2-i & for \quad \frac{n}{2}+2 \le i \le n, \end{cases} \quad \varphi(e_i) = \begin{cases} i & for \quad 1 \le i \le \frac{n}{2}, \\ n+1-i & for \quad \frac{n}{2}+1 \le i \le n, \end{cases}$$

$$\varphi(g_i) = \begin{cases} 2\left\lceil \frac{i}{2} \right\rceil & for \quad 1 \le i \le \frac{n}{2}-1, i odd, \\ i & for \quad 1 \le i \le \frac{n}{2}-1, i even, \\ \left\lceil \frac{2n+3}{4} \right\rceil & for \quad i = \frac{n}{2} \\ n+1-i & for \quad \frac{n}{2}+1 \le i \le n, \end{cases} \qquad \varphi(h_i) = \begin{cases} 1 & for \quad i = 1, \\ 3 & for \quad i = 2, \\ i+1 & for \quad 3 \le i \le \frac{n}{2} \\ n+2-i & for \quad \frac{n}{2}+1 \le i \le n, \end{cases}$$

Case 2: $k \equiv 1 (\bmod 4)$

$$\varphi(u_i) = \begin{cases} i & for \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\ n+1-i & for \quad \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \end{cases} \qquad \varphi(v_i) = \begin{cases} i & for \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{2n+3}{4} \right\rceil & for \quad i = \left\lceil \frac{n}{2} \right\rceil, \\ n+2-i & for \quad \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \end{cases}$$

$$\varphi(e_i) = \begin{cases} i & for \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\ n+1-i & for \quad \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \end{cases} \qquad \varphi(g_i) = \begin{cases} 2\left\lceil \frac{i}{2} \right\rceil & for \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, i odd, \\ i & for \quad 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, i even, \\ n+1-i & for \quad \left\lceil \frac{n}{2} \right\rceil \le i \le n, \end{cases}$$

$$\varphi(h_i) = \begin{cases} 1 & for \quad i = 1, \\ 3 & for \quad i = 2, \\ i+1 & for \quad 3 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ n+2-i & for \quad \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n, \end{cases}$$

Case 3: $k \equiv 2 (\bmod 4)$

$$\varphi(u_i)=\begin{cases} i & for \quad 1\le i\le \frac{n}{2}+1, \\ n+1-i & for \quad \frac{n}{2}+2\le i\le n, \end{cases}$$

$$\varphi(v_i)=\begin{cases} i & for \quad 1\le i\le \frac{n}{2}, \\ \left\lceil \frac{2n+3}{4} \right\rceil & for \quad i=\frac{n}{2}+1, \\ n+2-i & for \quad \frac{n}{2}+2\le i\le n, \end{cases}$$

$$\varphi(e_i)=\begin{cases} i & for \quad 1\le i\le \frac{n}{2}, \\ n+1-i & for \quad \frac{n}{2}+1\le i\le n, \end{cases}$$

$$\varphi(g_i)=\begin{cases} 2\left\lceil \frac{i}{2} \right\rceil & for \quad 1\le i\le \frac{n}{2}, iodd, \\ i & for \quad 1\le i\le \frac{n}{2}, ieven, \\ n+1-i & for \quad \frac{n}{2}+1\le i\le n, \end{cases}$$

$$\varphi(h_i)=\begin{cases} 1 & for \quad i=1, \\ 3 & for \quad i=2, \\ i+1 & for \quad 3\le i\le \frac{n}{2} \\ n+2-i & for \quad \frac{n}{2}+1\le i\le n, \end{cases}$$

Case 4: $k\equiv 3(\bmod 4)$

$$\varphi(u_i)=\begin{cases} i & for \quad 1\le i\le \left\lceil \frac{n}{2} \right\rceil, \\ n+1-i & for \quad \left\lceil \frac{n}{2} \right\rceil+1\le i\le n, \end{cases}$$

$$\varphi(v_i)=\begin{cases} i & for \quad 1\le i\le \left\lceil \frac{n}{2} \right\rceil, \\ n+2-i & for \quad \left\lceil \frac{n}{2} \right\rceil+1\le i\le n, \end{cases}$$

$$\varphi(e_i)=\begin{cases} i & for \quad 1\le i\le \left\lceil \frac{n}{2} \right\rceil, \\ n+1-i & for \quad \left\lceil \frac{n}{2} \right\rceil+1\le i\le n, \end{cases}$$

$$\varphi(g_i)=\begin{cases} 2\left\lceil \frac{i}{2} \right\rceil & for \quad 1\le i\le \left\lceil \frac{n}{2} \right\rceil, iodd, \\ i & for \quad 1\le i\le \left\lceil \frac{n}{2} \right\rceil, ieven, \\ n+1-i & for \quad \left\lceil \frac{n}{2} \right\rceil+1\le i\le n, \end{cases}$$

$$\varphi(h_i)=\begin{cases} 1 & for \quad i=1, \\ 3 & for \quad i=2, \\ i & for \quad 3\le i\le \left\lceil \frac{n}{2} \right\rceil-1, \\ \left\lceil \frac{2n+3}{4} \right\rceil & for \quad i=\left\lceil \frac{n}{2} \right\rceil, \\ n+1-i & for \quad \left\lceil \frac{n}{2} \right\rceil+1\le i\le n. \end{cases}$$

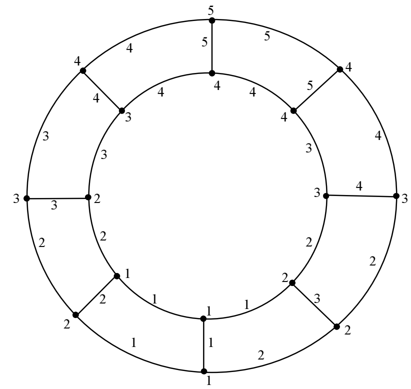


Figure 2: $tvs(CL(16))=5$

We observe that

$wt(u_1)=\varphi(u_1)+\varphi(e_1)+\varphi(e_n)+\varphi(h_1),\quad wt(v_1)=\varphi(v_1)+\varphi(g_1)+\varphi(g_n)+\varphi(h_1),$

For $2\le i\le n, wt(u_i)=\varphi(u_i)+\varphi(e_i)+\varphi(e_{i-1})+\varphi(h_i),\quad wt(v_i)=\varphi(v_i)+\varphi(g_i)+\varphi(g_{i-1})+\varphi(h_i).$

Hence

Case 1: $k\equiv 0(\mathrm{mod}\,4)$

$$wt(u_i)=\begin{cases}4i & for\quad 1\le i\le \frac{n}{2},\\ 4i-3 & for\quad i=\frac{n}{2}+1,\\ 5n-2-4i & for\quad \frac{n}{2}+1\le i\le n,\end{cases}\quad wt(v_i)=\begin{cases}4i+1 & for\quad 1\le i\le \frac{n}{2},\\ 4i+2 & for\quad i=\frac{n}{2},\\ 4i-1 & for\quad i=\frac{n}{2}+1,\\ 5n-1-4i & for\quad \frac{n}{2}+2\le i\le n,\end{cases}$$

Case 2: $k\equiv 1(\mathrm{mod}\,4)$

$$wt(u_i)=\begin{cases}4i & for\quad 1\le i\le \left\lceil\frac{n}{2}\right\rceil,\\ 5n-1-4i & for\quad \left\lceil\frac{n}{2}\right\rceil+1\le i\le n,\end{cases}\quad wt(v_i)=\begin{cases}4i+1 & for\quad 1\le i\le \left\lceil\frac{n}{2}\right\rceil,\\ 5n-2-4i & for\quad \left\lceil\frac{n}{2}\right\rceil+1\le i\le n,\end{cases}$$

Case 3: $k\equiv 2(\mathrm{mod}\,4)$

$$wt(u_i)=\begin{cases}4i & for\quad 1\le i\le \frac{n}{2},\\ 5n-4-4i & for\quad \frac{n}{2}+1\le i\le n,\end{cases}\quad wt(v_i)=\begin{cases}4i+1 & for\quad 1\le i\le \frac{n}{2},\\ 5n-3-4i & for\quad \frac{n}{2}+1\le i\le n,\end{cases}$$

Case 4: $k\equiv 3(\mathrm{mod}\,4)$

$$wt(u_i)=\begin{cases}4i & for\quad 1\le i\le \left\lceil\frac{n}{2}\right\rceil,\\ 5n+2-4i & for\quad \left\lceil\frac{n}{2}\right\rceil+1\le i\le n,\end{cases}\quad wt(v_i)=\begin{cases}4i+1 & for\quad 1\le i\le \left\lceil\frac{n}{2}\right\rceil,\\ 5n+3-4i & for\quad \left\lceil\frac{n}{2}\right\rceil+1\le i\le n,\end{cases}$$

See Figure 2. So the weights of the vertices of $CL(n)$ under the labeling φ constitute the set $\{4,5,6,7...,2n+3\}$ and the function φ is a mapping from $V(CL(n))\cup E(CL(n))$ into $\left\{1,2,3,...,\left\lceil\frac{2n+3}{4}\right\rceil\right\}$.

For $CL(1)$ we give the following special labeling: $\varphi(u_1)=1, \varphi(v_1)=1, \varphi(e_1)=1, \varphi(g_1)=2, \varphi(h_1)=1,$ For $CL(2)$ we give the following special labeling: $\varphi(u_1)=1, \varphi(u_2)=2, \varphi(v_1)=1, \varphi(v_2)=2, \varphi(e_1)=1,$ $\varphi(e_2)=1, \varphi(g_1)=2, \varphi(g_2)=1,$ $\varphi(h_1)=1, \varphi(h_2)=2,$ For $CL(3)$ we give the following special labeling: $\varphi(u_1)=1, \varphi(u_2)=1, \varphi(u_3)=2,$ $\varphi(v_1)=1,$ $\varphi(v_2)=1,$ $\varphi(v_3)=3,$ $\varphi(g_1)=2, \varphi(g_2)=2, \varphi(g_3)=1, \varphi(h_1)=1, \varphi(h_2)=2, \varphi(h_3)=3,$ For $CL(4)$ we give the special labeling: $\varphi(u_1)=1, \varphi(u_2)=2, \varphi(u_3)=2, \varphi(u_4)=1,$ $\varphi(v_1)=1, \varphi(v_2)=1, \varphi(v_3)=3, \varphi(v_4)=2,$ $\varphi(e_1)=1, \varphi(e_2)=2,$ $\varphi(e_3)=3,$ $\varphi(e_4)=1,$ $\varphi(g_1)=2, \varphi(g_2)=3, \varphi(g_3)=2, \varphi(g_4)=1,$ $\varphi(h_1)=2, \varphi(h_2)=3, \varphi(h_3)=3, \varphi(h_4)=2.$
It is easy to see that these total labelings have the required properties. This concludes the proof.

4. Windmill Graph

In this section we determine the total vertex irregularity strength of windmill graph $C_3^{(m)}$.

Definition 4.1[7]. The windmill graph $C_3^{(m)}$ is a family of graphs consisting of m copies of C_3 of with a vertex in common. See Figure 3 (a).

Theorem 4.1. $tvs(C_3^{(m)}) = \left\lceil \frac{2m+2}{3} \right\rceil$. See Figure 3.(b).

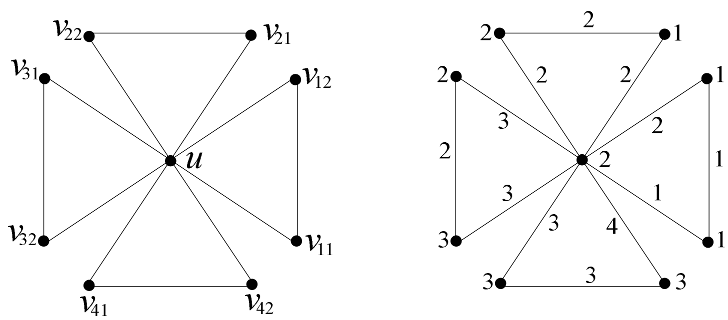


Figure 3 (a) $C_3^{(4)}$ **; (b)** $tvs(C_3^{(4)}) = 4$

5. Conclusion In this paper we have determined the total vertex irregularity strength of circular ladder and windmill graphs. Total vertex irregular k-labeling for networks like hexagonal network, butterfly network and benes network is under investigation.

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