Optimal book embedding of the generalized Petersen graph $P(n, 2)$

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Abstract

A *book* consists of a line in the 3-dimensional space, called the spine, and a number of *pages*, each a half-plane with the spine as boundary. A *book embedding* (π, ρ) of a graph consists of a linear ordering of π, of vertices, called the *spine ordering*, along the spine of a book and an assignment ρ, of edges to pages so that edges assigned to the same page can be drawn on that page without crossing. That is, we cannot find vertices u, v, x, y with $\pi(u) < \pi(x) < \pi(v) < \pi(y)$, yet the edges uv and xy are assigned to the same page, that is $\rho(uv) = \rho(xy)$. The *book thickness* or *page number* of a graph G is the minimum number of pages required to embed G in a book. In this paper we consider the Generalized Petersen Graphs and obtain the printing cycle for embedding them in a book. We prove that the Generalized Petersen graph $P(n, 2)$ can be embedded in three pages for any n. We also give a linear time algorithm to embed the Generalized Petersen graphs in three pages of a book.

Key words: Book embedding, book thickness, page number, VLSI design

1. Introduction

The growth of the subject 'graph theory' has been very rapid in recent years, particularly since the domain of its application is extremely varied. Graph algorithms play a very important role in design of various computer networks. Among the problems one comes across in graph theory, is the embedding of graphs. A particular way of embedding graphs is in the pages of a book. The book embedding of graphs was first introduced by Bernhart and Kainen [1] and since then, many researchers have actively studied it. Determining the book thickness for general graphs is *NP*-hard. But obtaining the book thickness for particular graphs have been found to be possible. The book embeddings have been studied for many classes of graphs. To name a few, we have: Complete Graphs [1, 2], Complete Bipartite Graphs[10], Trees, Grids and X-trees [3], hypercubes [3, 9], incomplete hypercubes [8], iterated line digraphs [6], de Bruijn graphs, Kautz graphs, shuffle-exchange graphs [7], for each of which embedding in books have been studied.

The book embedding problem has many different applications, which include sorting with parallel stacks, single-row routing of printed circuit boards, and the design of fault-tolerant processor arrays [4, 12].

2. Preliminaries

The Petersen graph is a popular network, having fixed valance, small diameter and several other optimal properties. Several network topologies have been proposed based on the Petersen graph, and many of them can be found in [11]. In [13],Watkins introduced the notion of *generalized Petersen graph* (*GPG* for short).

Definition 1: Given integers $n \geq 3$ and $k \in Z_n - \{0\}$, the generalized Petersen graph $P(n, k)$ is defined on the set $\{x_i, y_i| \ i \in Z_n\}$ of $2n$ vertices, with the adjacencies given by x_ix_{i+1}, x_iy_i, y_iy_{i+k} for all i. See Figure 1. The generalized Petersen graph $P(n, k)$ is nonplanar for $n \geq 5$ and $k \geq 2$.

3. Embedding of the generalized Petersen graph $P(n, 2)$

We shall now discuss the pagenumber of the generalized Petersen graph $P(n, 2)$. We shall consider n larger than 5.

Lemma 2: The lower bound for the pagenumber of the generalized Petersen graph $P(n, 2)$, $n \geq 5$ is 3.

Proof: We first note that $P(n, 2)$ is nonplanar for $n \geq 5$. Hence it cannot be embedded in two pages [1]. Therefore the minimum pagenumber for $P(n, 2)$ is 3, which is the required lower bound. The lemma is proved. ∎

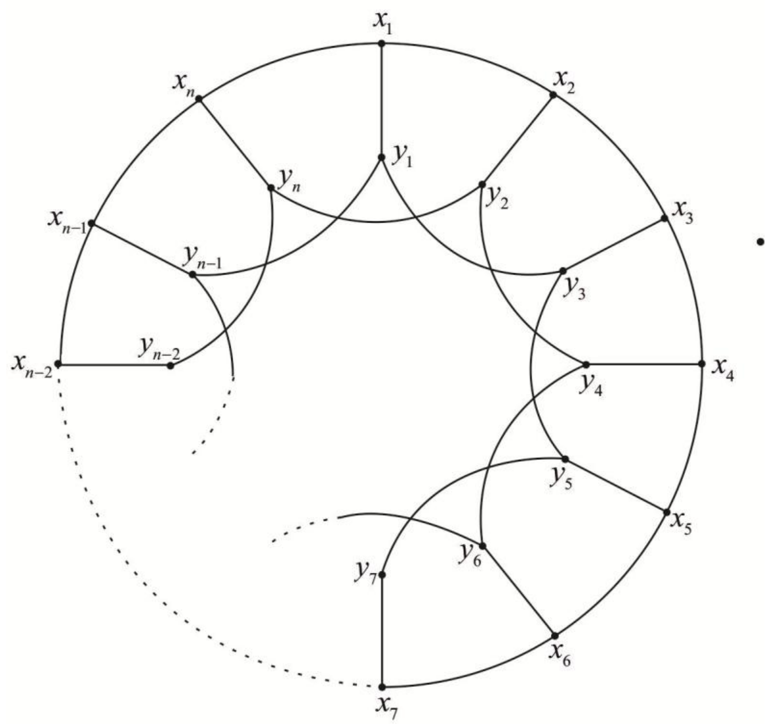
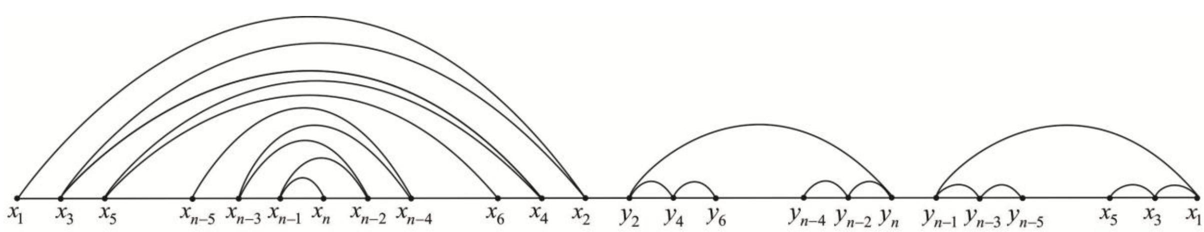


Figure 1 Generalized Petersen graph $P(n, k)$

Notation

Let $X = \{x_1, x_2, ..., x_n\}$ be any sequence. We denote by X_o, the sequence $\{x_i : i \text{ is odd}\}$. Similarly X_e denotes the series $\{x_i : i \text{ is even}\}$. The Sequence $\{x_n, x_{n-1}, ..., x_1\}$ is denoted by X^R.



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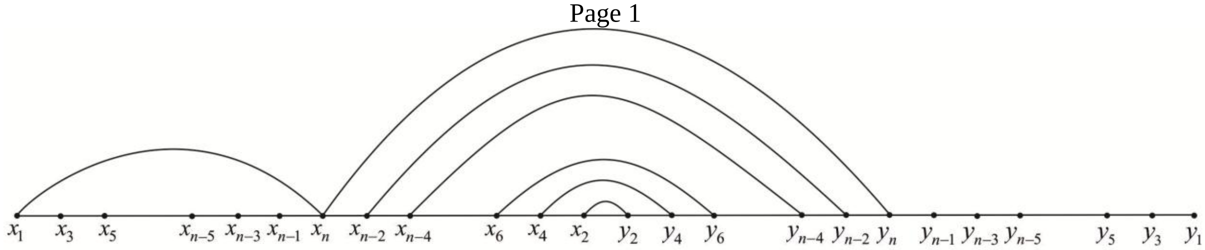

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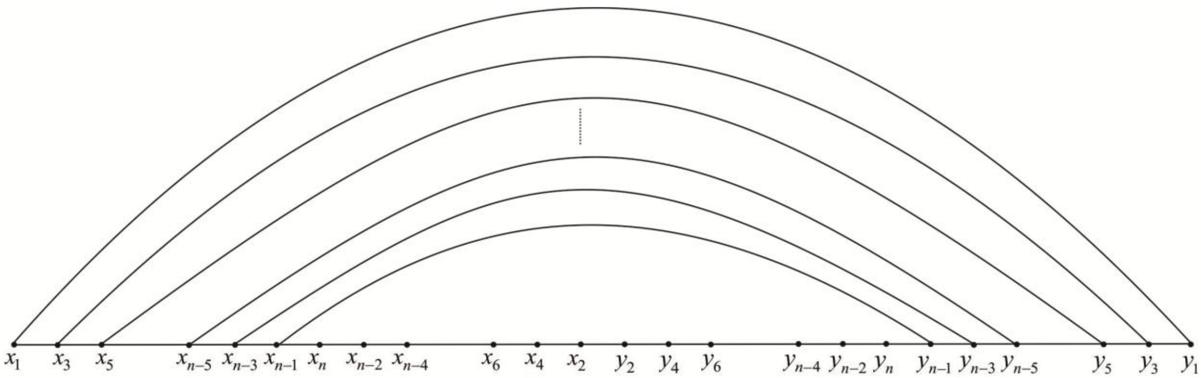


Page 3

Figure 2 Three page embedding of the Generalized Petersen graph $P(n, 2)$ for even n

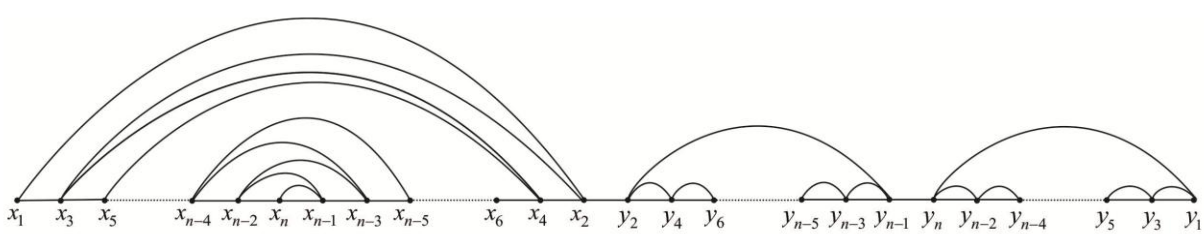
Theorem 3: The generalized Petersen graph $P(n, 2)$ can be embedded in three page for all n.

Proof: By Lemma 2, the minimum number of pages required by the graph $P(n, 2)$ is 3. We shall now provide a scheme for embedding $P(n, 2)$ in 3 pages for all n.

Proof: Let the vertices $P(n, 2)$ be the set $\{x_i, y_i \,/\, i \in Z_n\}$ with edges $x_i x_{i+1}, x_i y_i, y_i y_{i+2}$, for each $i = 1, ..., n$. Let $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_n\}$. For the printing cycle, consider the sequence, $X_o X_e^R Y_e Y_o^R$. With the above sequence marked on the spine, in the same order, embed the edges of $P(n, 2)$ in pages 1, 2 and 3 as follows.

$$\text{Embed the edge } \left\{ \begin{array}{l} x_i x_{i+1}; 1 \le i \le n-1 \text{ in page 1} \\ y_i y_{i+2}; 1 \le i \le n-2 \text{ in page 1} \\ \mathrm{y}_2 y_n \text{ in page 1} \\ x_1 x_n \text{ in page 2} \\ \mathrm{y}_1 y_{n-1} \text{ in page 2} \\ x_i y_i; i \text{ is even; in page 2} \\ x_j y_j; j \text{ is odd in page 3} \end{array} \right\}. \text{ See Figure 2, 3.}$$

The above embedding scheme would embed $P(n, 2)$ in three pages. ■



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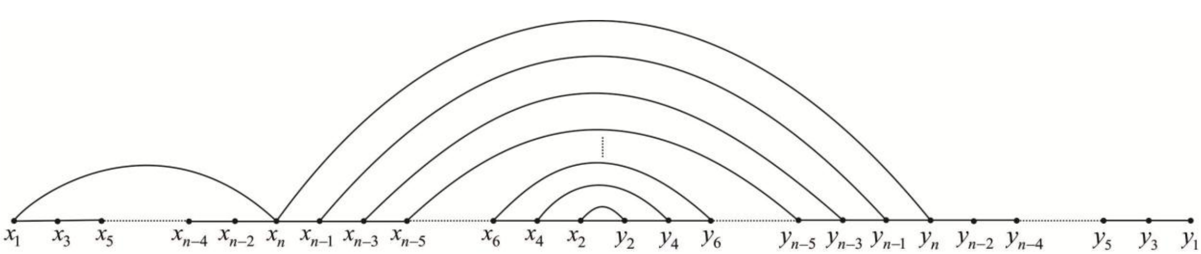

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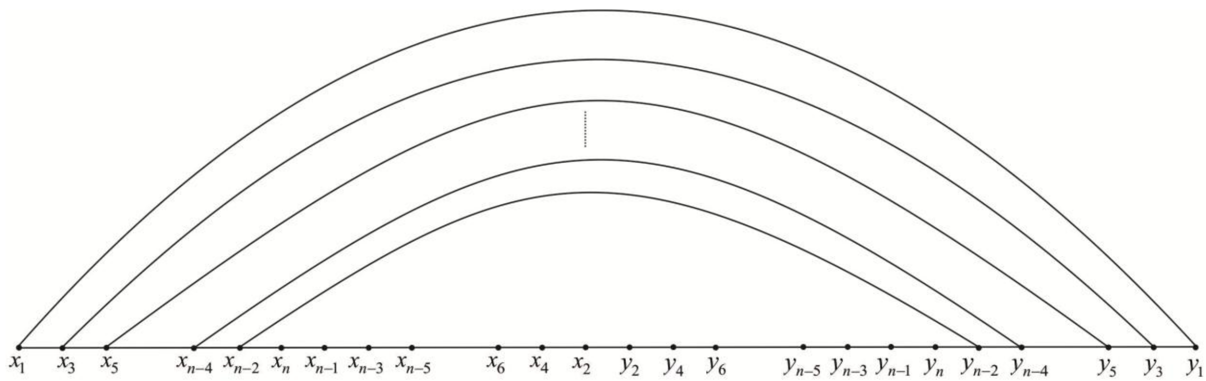

Page 3

Figure 3 Three page embedding of the Generalized Petersen graph *P*(*n*, 2) for odd n

4. Embedding Algorithm

In what follows, we present the embedding algorithm to embed the generalized Petersen graph *P*(*n*, 2) in three pages.

1 Algorithm *BkEmbedGPG*(*a*, *n*)
2 {
3 // The algorithm embeds the generalized Petersen graph P(*n*, 2) in three pages, using the
4 // printing cycle $X_o X_e^R Y_e Y_o^R$.
5 // For $i = 1, 2, \ldots, n$, $a[i]$ gives the page in which the edge $(x_i, x_{(i+1) \bmod n})$
6 // gets embedded. For $i = n+1, \ldots, 2n$, $a[i]$ gives the page in which the edge
7 // $(y_j\, y_{(j+2) \bmod n})$; $j = 1, \ldots, n$ gets embedded. For $i = 2n+1, \ldots, 3n$, $a[i]$ gives the page
8 // in which the edges (x_j, y_j); $j = 1, \ldots, n$ gets embedded
9// Embedding of edges (x_i, x_j)
10 for(j = 1; j < n; j++)
11 $a[j]$ = 1;
12 $a[n]$ = 2;
13 //Embedding of the edges (y_i, y_j)
14 for(j = 1; j<= n-2; j++)
15 $a[n+j]$ = 1
16 $a[2*n-1]$ = 2;
17 $a[2*n]$ = 1;
18 //Embedding the edges (x_i, y_i)
19 for(j = 2; j<=2*$\lfloor n/2 \rfloor$;j++)
20 $a[j]$ = 2;
21 for(j = 1 ; j<=2*$\lfloor (n-1)/2 \rfloor$;j++)
22 $a[j]$ = 3;
23 }

Note 1

Let us now give a shout account o*f Algorithm BkEmbedGPG*(*a*, *n*).

The *for* loop in lines 10 and 11 assigns $a[1]$, ..., $a[n-1]$ with the value 1. This means that the edges (x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n) are embedded in page 1. Line 12 embeds (x_n, x_1) in page 2.

The *for* loop in lines 14 and 15 assigns $a[n+1]$, ..., $a[2*n-2]$ with the value 1. This would mean that the edges (y_1, y_3), (y_2, y_4), ..., (y_{n-2}, y_n) in page 1. Line 16 embeds (y_{n-1}, y_1) in page 2 and line 17 embeds (y_n, y_2) in page 1.

The *for* loop in lines 19 and 20 assigns *a*[2**n*+2], *a*[2**n*+4], ... with the value 2. This would embed the edges (x_2, y_2), (x_4, y_4), ... in page 2.

The *for* loop in lines 21 and 22 assigns *a*[2**n*+1], *a*[2**n*+3], ... with the value 3. This would embed the edges (x_2, y_2), (x_4, y_4), ... in page 3.

Thus the algorithm embeds all the edges of *P*(*n*, 2) in three pages.

5. Proof of correctness of Embedding Algorithm

Theorem 4: Algorithm *BkEmbedGPG*(*a*, *n*) embeds the generalized Petersen graph *P*(*n*, 2) in three pages for any *n*.
Proof: We see that by Note 1, Algorithm *BkEmbedGPG*(*a*, *n*) embeds the edges of *P*(*n*, 2) according to the scheme given in Theorem 2. Hence it embeds the edges of *P*(*n*, 2) in three pages of a book. ■

6. Time Complexity

Having given the proof of correctness, we shall prove that Algorithm *BkEmbedGPG*(*a*, *n*) is a linear time algorithm.

Theorem 5: Algorithm *BkEmbedGPG*(*a*, *n*) executes in $\Theta(n)$ time.

Proof: It is enough to prove that Algorithm *BkEmbedGPG*(*a*, *n*) requires *O*(*n*) time to execute.
Lines 10 & 11 include the range of *for* loop which executes in *O*(*n*) time. Line 12 executes in *O*(1) time. Lines 14 & 15 executes in *O*(*n*) time. Lines 16 & 17 executes in *O*(1) time. Lines 19 to 22 works in *O*(*n*) time as the *for* loop ranges in lines 19 & 20 and 21 & 22 does so. In all the algorithm executes in *O*(*n*) time. ■

7. Conclusion

The discussion in section 3 shows that the embedding scheme suggested in Theorem 2 is optimal with respect to the number of pages. Further Theorem 4 in section 6 shows that the algorithm *BkEmbedGPG*(*a*, *n*) is optimal with respect to time. We also see that each page has a page width of *O*(*n*), which more or less evenly distributes the 3*n* edges of *P*(*n*, 2) in three pages.

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